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I.—ON A DIFFICULTY IN THE THEORY OF ALGEBRA.

By D. F. Gregory, M.A. Fellow of Trinity College.

I ought perhaps to apologize to the reader for calling his attention to a subject so much discussed as the nature of the symbols + and − when used in symbolical Algebra. But the theory which I wish to develop appears to me to remove some part of the difficulties which, after all which has been written, still adhere to the subject; and I am the more anxious to explain my views, because I have in previous papers held, in common I believe with every other writer, an opinion which a more attentive consideration induces me to think erroneous. It is generally assumed that the symbols + and − signify primarily addition and subtraction, and that any other meanings which we may attach to them must be derived from the fundamental significations. The theory which I have now to maintain is the apparently paradoxical one, that the symbols + and − do not represent the arithmetical operations of addition and subtraction; and that though they were originally intended to bear these meanings, they have become really the representatives of very different operations. This is opposed to our preconceived ideas, but I trust that the following statements will show the truth of the assertion.

When it is said that any symbol does not represent a particular operation, it is necessary to explain what is to be understood by an *Algebraical* symbol, and in what way it represents an operation. In previous papers on the Theory of Algebra, I have maintained the doctrine that a symbol is defined *algebraically* when its laws of combination are given; and that a symbol represents a given operation when the laws

of combination of the latter are the same as those of the former. This, or a similar theory of the nature of Algebra, seems to be generally entertained by those who have turned their attention to the subject: but without in any degree leaning on it, we may say that symbols are actually subject to certain laws of combination, though we do not suppose them to be so defined; and that a symbol representing any operation must be subject to the same laws of combination as the operation it represents. These assertions are independent of any theory except this, that there is a general Symbolical Algebra different from Arithmetical Algebra, in which the laws of the combination of the symbols are attended to. When therefore I say that the symbol $+$ does not represent the arithmetical operation of addition, or $-$ that of subtraction, I mean that the laws of combination of the symbols $+$ and $-$ are not those of the operations of addition and subtraction. Now the laws of combination of the symbols $+$ and $-$ are four, viz.

$$++a=+a,\quad +-a=-a,\quad -+a=-a,\quad --a=+a;$$

and whatever may be our theory of the meaning of these symbols, there is no doubt that we always assume them to be subject to these laws, which are indeed the very first steps which the student in algebra makes. But no one who considers the nature of the arithmetical operations can hesitate for a moment to say, that they are not subject to these laws; of which a sufficient proof is this, that addition and subtraction are inverse operations, whereas the second and third of the preceding laws are inconsistent with the idea that $+$ and $-$ are inverse symbols, the character of which is, that the one undoes what the other does; so that if f, ϕ are two symbols representing inverse operations, we have

$$f\phi(a)=a \quad \text{and} \quad \phi f(a)=a.$$

The same conclusion also follows from the fourth law, which is evidently analogous to the relation between a number and its square, not to that between a number and its reciprocal. A natural objection which may be brought against the view which I am here maintaining is, that we do actually define $+$ and $-$ to be the symbols representing addition and subtraction, and therefore that they must represent these operations. A further examination however will shew that this is not the case. We say that $a+x$ is to represent x added to a, and $a-x$, x subtracted from a: we do not directly assert that $+$ signifies addition, and $-$ subtraction. If we did we should contradict ourselves, when we asserted that $+-a=-a$, or $--a=+a$. The fact is, that we are deceived by writing

a *sum* and *difference* in a manner different from that in which we express the performance of any other operation. When we wish to denote that an operation f is performed on a subject a, we usually prefix the symbol of operation to the subject, and write $f(a)$: this however is not necessary, for we might connect them in any other way; and indeed Mr. Murphy prefixes the subject to the symbol of operation, apparently for the purpose of avoiding the prejudices which our ordinary mode of writing is apt to produce. But, on the other hand, when we wish to denote the addition of a number x to a number a, or its subtraction from it, we write

$$a + x \text{ or } a - x,$$

the operations being indicated by writing the symbol of the number added or subtracted *after* the subject, and separated from it by certain symbols; whereas in the ordinary mode of writing we should *prefix* the operating symbol. Now if we merely assert that $a + x$ is to signify the addition of x to a, the symbol + might be understood to *indicate* addition, it being used to distinguish the kind of operation involving x which is performed on a, in the same way as $a \times x$ indicates the multiplication of a by x. But by such an assertion we do not make + an *algebraical* symbol in the sense in which I use the word; nor does it *represent* the operation, though it may indicate it. It is only when we arrive at such conclusions as $a + (x + y) = a + x + y$, involving the law $+ + a = + a$, that we give to + an algebraical individuality as a symbol subject to certain laws of combination, which, we see at once, are not those belonging to the operation of addition. If on this any one chooses to say that he considers + as indicating (he cannot say *representing*) the operation of addition, and that he does not trouble himself about its laws of combination, there can be no objection to his holding such an opinion except this, that the Algebra in which such a symbol is used is not a general science, but simply Arithmetic. He cannot, consistently with this doctrine, hold that direction in Geometry can be indicated by + and −; or that these symbols can receive any other interpretation than that which was originally assigned to them: conclusions inconsistent with any conceptions which we can form of a *general* Algebra. There is no doubt that we *can* give these symbols such a geometrical interpretation, and it is the possibility of so doing which has occasioned the difficulties of the Theory of Algebra considered as something more general than Arithmetic, and which has led to the more extended views which in recent years have been taken of the subject.

The preceding observations may be illustrated by using new symbols to represent the operations of addition and subtraction, by prefixing them in the ordinary way to the subject, and so investigating their laws of combination. Let us assume the symbol A to be that which represents addition, and B that of subtraction, and let us attach to them as a suffix the quantity which is added or subtracted ; so that

$$A_x(a) \text{ and } B_x(a)$$

represent the addition of x to a and the subtraction of x from a. One of the most obvious laws of the operation of addition is, that if x and y be two quantities which are added to a, it is indifferent in what order the operations are performed: so that if we first add x to a, and then y to the sum, we obtain the same result as if we first added y to a, and then x to the sum. This law is expressed by the equation

$$A_xA_y(a) = A_yA_x(a);$$

that is, the operations A_x, A_y are commutative.

Again: another law is, that each of these sums is the same as if y were first added to x, and then that sum added to a. Now the sum of y and x is represented by $A_y(x)$, and therefore the addition of this sum to a is represented by $A_{A_y(x)}(a)$; so that the law in question is expressed by the equation

$$A_xA_y(a) = A_{A_y(x)}(a).$$

The same law may be extended to any number of variables x, y, z, so that we have the equation

$$A_xA_yA_z = A_{A_{A_z(y)}(x)}(a), \text{ and so on.}$$

The notation is an inconvenient one: but it is not here introduced for the purpose of supplanting the common one, but merely to show how the laws of combination of the operation of addition may be represented by one symbol only, without the aid of a subsidiary symbol such as $+$.

A third law of the operation of addition is, that it is indifferent whether x be added to a, or a to x: this is expressed by the equation

$$A_x(a) = A_a(x).$$

These three laws of combination are sufficient for our purpose, and show distinctly in a symbolical form how different the laws of the operation of addition are from the laws of combination of the symbol $+$, which therefore cannot represent it.

With regard to subtraction, since it is the operation which is inverse to addition, we have plainly

$$A_x B_x (a) = B_x A_x (a) = a.$$

Also, without going into detail, it is easy to see that

$$B_x A_y (a) = A_{B_x(y)}(a) = B_{B_y(x)}(a);$$

and thus the laws of the operation of subtraction are represented by means of the symbols of addition and subtraction.

By means of this notation we see distinctly how the symbol $+$ appears frequently as a *separative* symbol between two others, in such a way that the order of the symbols cannot be changed. Thus we cannot say $+ax$ instead of $a + x$, though we do say that $a\sqrt{-1}\,x$, or $a(-)^{\frac{1}{2}}x$, is equivalent to $(-)^{\frac{1}{2}}ax$: for $a + x$ is equivalent to $A_x(a)$, whereas $a(-)^{\frac{1}{2}}x$ signifies only the successive performance of these operations which are commutative; the symbol $\sqrt{-1}$, or $(-)^{\frac{1}{2}}$, not having acquired the double signification which is attached in consequence of their position, to $+$ or $-$. In the same way, though we say that $a + (-x) = a - x$, we do not say that $a + \sqrt{-1}\,x$, or $a + [(-)^{\frac{1}{2}}x] = a(-)^{\frac{1}{2}}x$, because these two formulæ are equivalent to $A_{-x}(a)$ and $A_{(-)^{\frac{1}{2}}x}(a)$, and the law that $A_{-x}(a) = B_x(a)$ has no analogue in the case of $(-)^{\frac{1}{2}}$, or other powers of $+$ and $-$.

The distributive law, which is met with so frequently in algebraical operations, and which is usually written

$$f(a + x) = f(a) + f(x),$$

is in this notation expressed by the equation

$$f[A_x(a)] = A_{f(x)}[f(a)].$$

In like manner the index law, which is usually written

$$f^m f^n (a) = f^{m+n}(a),$$

becomes in this notation

$$f^m f^n (a) = f^{A_n(m)}(a).$$

To one who is acquainted with the higher branches of mathematics it is obvious that the operation, which is here denoted by A, is the same in kind as that of which so much use is made in the Calculus of Finite Differences for converting $f(x)$ into $f(x + h)$, and which has been represented by

the symbol D^k in the papers on the subject in preceding numbers of this journal.

Before concluding I would say a few words on what appears to me to be a prejudice relative to the nature of the symbols + and −. These are generally considered to be absolutely distinct from literal symbols, and have in consequence a different name assigned to them, being called "signs of affection." Such a distinction exists in arithmetical, but not in general Algebra. In the former, literal symbols are used to represent numbers or magnitudes, and are capable of receiving interpretation, that is, of having different meanings or values assigned to them, while the signs of affection indicate the performance of certain operations, and are incapable of bearing any other meaning than those which are originally assigned to them. As such, the signs + and − are exactly on a par with × and ÷, though the latter, from accidental circumstances, have not become so important as the former. When we write the symbol a in Arithmetical Algebra, we mean that we may substitute for it any number we choose; but when we write $a + b$, we say that b is added to a, we attach to + a definite meaning, and we can give no other interpretation to it, without taking into consideration its laws of combination, which are excluded from Arithmetical Algebra. On the other hand, in Symbolical Algebra, where every symbol represents an operation, it is obvious that we cannot *à priori* speak of any difference in kind between different symbols. In such a science what is a, and what is +? To neither are definite meanings attached, as to the latter symbol in Arithmetical Algebra. Our conceptions would be clearer, and our minds more free from prejudice, if we never used in the general science symbols to which definite meanings had been appropriated in the particular science. Inveterate practice has however so wedded us to the use of the symbols + and − that we find it difficult to dispense with them, and still more difficult, in using them, to avoid being misled by ideas drawn from Arithmetic. The symbols + and −, × and ÷, were invented for the purpose of indicating the performance of certain operations on numbers; but as the science advanced, it was found that the symbol × might be conveniently omitted, the operation being indicated merely by the juxtaposition of symbols; so that ax stood for $a \times x$. From this the transition was easy to the conception of a as the symbol of the operation; a change of great importance, as leading to the view that Symbolical Algebra is a Calculus of Operations. But it is merely a matter of accident that the symbol × was that which was expunged: that fate might as

well have befallen the symbol +, and then $a\,x$ would have signified the addition of a to x, and the difficulties which have been experienced regarding + and − would then have been transferred to × and ÷. It is perhaps, to a certain extent, unfortunate that we have in multiplication represented by one letter the symbol of the operation as well as that with respect to which it is performed: the latter ought rather to be attached as an index or a suffix. Thus, if we represented the multiplication of x by a by the symbol $P_a(x)$, we should have no difficulty in seeing that it was exactly analogous to addition under the notation $A_a(x)$, as I have in this paper written it.

In the preceding remarks I have proceeded on the supposition that Symbolical Algebra must be considered as a science of operations represented symbolically: this view may not appear to every one necessary; but if the subject be considered in all its generality, it will, I am convinced, be found that there is no other way of explaining the difficulties of Algebra in a uniform and consistent manner.

II.—ON THE LIMITS OF MACLAURIN'S THEOREM.

By A. Q. G. Craufurd, M.A. Jesus College.

To the Editor of the Cambridge Mathematical Journal.

Sir,—There is an error in my last paper which I did not observe till I saw it in print, and which I now hasten to correct. But before doing so, I must give the reader notice, that in what follows I shall use the symbol $\underset{a^n}{C}$ instead of $\overset{a}{\underset{n}{C}}$. The former is more analogous than the latter to the ordinary notation of algebra, and may be usefully extended by taking $\underset{e^{nx}}{C}$ to represent the coefficient of e^{nx} in a series of powers of e^x; and $\underset{\cos na}{C}$ for the coefficient of $\cos na$ in a series of the cosines of multiples of a.

The error to which I have alluded occurs in page 85, line 10, where, having shewn that the terms of Maclaurin's series which follow that affected with x^n, may be represented by

$$x^{n+1}\underset{a^{n+1}}{C}\cdot\frac{f(a)}{1-\dfrac{x}{a}},\ \left(\text{or } x^{n+1}\underset{a^n}{C}\cdot\frac{f(a)}{a-x},\right)$$

I proceed to substitute for this expression

$$\frac{x^{n+1}}{1.2\ldots.(n+1)}\cdot\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(a)}{1-\dfrac{x}{a}}\right\}_{a=0},$$

which is the same as

$$\frac{x^{n+1}}{1.2\ldots.n}\cdot\left\{\frac{d^n}{da^n}\,\frac{f(a)}{a-x}\right\}_{a=0}.$$

I was led to this conclusion by observing, that the quantity $\dfrac{f(a)}{a-x}$ *may be* developed in a series of positive powers of a; and from this I inferred, that the coefficient of a^n in its development must be equal to $\dfrac{1}{1.2\ldots.n}\times$ (the value corresponding to $a=0$ of its n^{th} differential coefficient). But this argument is fallacious, for the expression $\underset{a^n}{C}.\dfrac{f(a)}{a-x}$ is indeterminate; it has at least two values, and

$$\frac{1}{1.2\ldots.n}\cdot\left\{\frac{d^n}{da^n}\cdot\frac{f(a)}{a-x}\right\}_{a=0}$$

is one of these values, but not the right one. Supposing $f(a)$ to be developable only in a series of positive powers of a, the values of $\underset{a^n}{C}.\dfrac{f(a)}{a-x}$ are

$$\underset{a^{n+1}}{C}f(a)+x\underset{a^{n+2}}{C}f(a)+x^2\underset{a^{n+3}}{C}f(a)+\&\text{c. to infinity},$$

and $-\{x^{-1}\underset{a^n}{C}f(a)+x^{-2}\underset{a^{n-1}}{C}f(a)+x^{-3}\underset{a^{n-2}}{C}f(a)\ldots.x^{-(n+1)}\underset{a^0}{C}f(a)\}.$

The former is that which results from developing $\dfrac{1}{a-x}$ in a series of negative powers of a; the latter results from developing the same in a series of positive powers. If each of these series be multiplied by x^{n+1}, the former gives those terms of Maclaurin's series which follow that affected with x^n (or *the remainder*); the latter, which is equal to

$$\frac{1}{1.2\ldots.n}\cdot\left\{\frac{d^n}{da^n}\cdot\frac{f(a)}{a-x}\right\}_{a=0},$$

gives all the terms up to that affected with x^n, with their signs changed, which are equivalent to the remainder $-f(x)$.

Consequently, the only real results of my paper in the 14th number of the Journal, are those which give the sum of any

part of the series by means of the symbol C_{a^n}, and those which give the sums of a finite number of terms by means of the symbol of differentiation; the latter may be written thus.

The terms of Maclaurin's series from that affected with x^n to that affected with x^m, both included, are equivalent to

$$\frac{x^n}{1.2\ldots.m}\left\{\frac{d^m}{da^m}\cdot f(a)\cdot\frac{x^{m-n+1}-a^{m-n+1}}{x-a}\right\}_{a=0}.$$

When $n=0$, this becomes

$$\frac{1}{1.2.3\ldots.m}\left\{\frac{d^m}{da^m}\cdot f(a)\cdot\frac{x^{m+1}-a^{m+1}}{x-a}\right\}_{a=0},$$

$$=\frac{x^{m+1}}{1.2.3\ldots.m}\left\{\frac{d^m}{da^m}\cdot\frac{f(a)}{x-a}\right\}_{a=0};$$

which is the value of all the terms up to that affected with x^m

The corresponding expressions for Taylor's series are,

$$\frac{h^n}{1.2\ldots.m}\left\{\frac{d^m}{da^m}\cdot f(x+a)\cdot\frac{h^{m-n+1}-a^{m-n+1}}{h-a}\right\}_{a=0},$$

and
$$\frac{1}{1.2\ldots.m}\left\{\frac{d^m}{da^m}\cdot f(x+a)\cdot\frac{h^{m+1}-a^{m+1}}{h-a}\right\}_{a=0},$$

$$=\frac{h^{m+1}}{1.2.3.\ldots.m}\left\{\frac{d^m}{da^m}\cdot\frac{f(x+a)}{h-a}\right\}_{a=0}.$$

Cor. The portions of the two series intermediate between the terms affected with the m^{th} and n^{th} powers (but including them), may also be represented by

$$\frac{x^{m+1}}{1.2\,.\,.\,m}\left\{\frac{d^m}{da^m}\cdot\frac{f(a)}{x-a}\right\}_{a=0}-\frac{x^n}{1.2\,.\,.\,(n-1)}\left\{\frac{d^{n-1}}{da^{n-1}}\cdot\frac{f(a)}{x-a}\right\}_{a=0},$$

and

$$\frac{h^{m+1}}{1.2\,.\,.\,m}\left\{\frac{d^m}{da^m}\cdot\frac{f(x+a)}{h-a}\right\}_{a=0}-\frac{h^n}{1.2\,.\,.\,(n-1)}\left\{\frac{d^{n-1}}{da^{n-1}}\cdot\frac{f(x+a)}{h-a}\right\}_{a=0};$$

which expressions are easily reduced to those previously given.

London, April 6th, 1842.

III.—ON CERTAIN EXPANSIONS, IN SERIES OF MULTIPLE SINES AND COSINES.

By Arthur Cayley, B.A. Trin. Coll.

In the following paper we shall suppose ε the base of the hyperbolic system of logarithms; e a constant, such that its modulus, and also the modulus of $\frac{1}{e}\{1-\surd(1-e^2)\}$, are each of them less than unity; $\chi\{\varepsilon^{u\surd(-1)}\}$ a function of u, which, as (u) increases from 0 to π, passes continuously from the former of these values to the latter, without becoming a maximum in the interval, $f(\varepsilon^{u\surd(-1)}$ any function of (u) which remains finite and continuous for values of u included between the above limits. Hence, writing

$$\chi\{\varepsilon^{u\surd(-1)}\}=m \; \ldots\ldots\ldots\ldots (1),$$

and considering the quantity

$$\frac{\surd(1-e^2).f\{\varepsilon^{u\surd(-1)}\}}{\surd(-1)\,\chi'\{\varepsilon^{u\surd(-1)}\}\,(1-e\cos u)} \ldots\ldots (2),$$

as a function of m, for values of m or u included between the limits 0 and π, we have

$$\frac{\surd(1-e^2).f\{\varepsilon^{u\surd(-1)}\}}{\surd(-1)\,\chi'\{\varepsilon^{u\surd(-1)}\}(1-e\cos u)}$$

$$=\frac{2}{\pi}\Sigma_{-\infty}^{\infty}\cos rm\int_0^\pi \frac{\surd(1-e^2).f\{\varepsilon^{u\surd(-1)}\}\cos rm\,dm}{\surd(-1)\,\chi'\{\varepsilon^{u\surd(-1)}\}\,(1-e\cos u)} \ldots\ldots (3),$$

(Poisson, *Mec.* tom. I. p. 650); which may also be written

$$\frac{\surd(1-e^2).f\{\varepsilon^{u\surd(-1)}\}}{\surd(-1)\,\chi'\{\varepsilon^{u\surd(-1)}\}\,(1-e\cos u)}$$

$$=\frac{2}{\pi}\Sigma_{-\infty}^{\infty}\cos rm\int_0^\pi \frac{\surd(1-e^2).f\{\varepsilon^{u\surd(-1)}\}\cos r\,\chi\{\varepsilon^{u\surd(-1)}\}\,du}{1-e\cos u} \ldots (4).$$

And if the first side of the equation be generally expansible in a series of multiple cosines of m, instead of being so in particular cases only, its expanded value will always be the one given by the second side of the preceding equation.

Now, between the limits 0 and π, the function

$$f\{\varepsilon^{u\surd(-1)}\}\cos r\,\chi\{\varepsilon^{u\surd(-1)}\}$$

will always be expansible in a series of multiple cosines of u; and if by any algebraical process the function $f\rho\cos r\chi\rho$ can be expanded in the form

$$f\rho\cos r\chi\rho=\Sigma_{-\infty}^{\infty}a_s\rho^s, \quad (a_s=a_{-s}) \ldots\ldots (5);$$

we have, in a convergent series,

$$f\{\varepsilon^{u\sqrt{(-1)}}\} \cos r\chi \{\varepsilon^{u\sqrt{(-1)}}\} = a_0 + 2\Sigma_1^\infty a_s \cos su \ldots (6).$$

Again, putting $\frac{1}{e}\{1 - \sqrt{(1-e^2)}\} = \lambda \ldots\ldots\ldots\ldots\ldots (7),$

we have $$\frac{\sqrt{(1-e^2)}}{1 - e\cos u} = 1 + 2\Sigma_1^\infty \lambda^p \cos pu \ldots\ldots\ldots (8).$$

Multiplying these two series, and effecting the integration, we obtain

$$\frac{1}{\pi}\int_0^\pi \frac{\sqrt{(1-e^2)}\,.\,f\{\varepsilon^{u\sqrt{(-1)}}\}\cos r\chi\{\varepsilon^{u\sqrt{(-1)}}\}du}{1 - e\cos u} = 2\{\tfrac{1}{2}a_0 + \Sigma_1^\infty(a_s\lambda^s)\} \ldots (9).$$

And the second side of this equation being obviously derived from the expansion of $f\lambda \cos r\chi\lambda$, by rejecting negative powers of λ, and dividing by 2 the term independent of λ, may conveniently be represented by the notation

$$\underbrace{2f\lambda\cos r\chi\lambda} \ldots\ldots\ldots\ldots (10);$$

where in general, if $\Gamma\,.\,\lambda$ can be expanded in the form

$$\Gamma\,.\,\lambda = \Sigma_{-\infty}^\infty (A_s\lambda^s), \quad [A_{-s} = A_s] \ldots (11),$$

we have $$\underbrace{\Gamma\,.\,\lambda} = \tfrac{1}{2}A_0 + \Sigma_1^\infty A_s\lambda^s \ldots\ldots (12).$$

(By what has preceded, the expansion of $\Gamma\,.\,\lambda$ in the above form is always possible in a certain sense; however, in the remainder of the present paper, $\Gamma\,.\,\lambda$ will always be of a form to satisfy the equation $\Gamma\,.\left(\frac{1}{\lambda}\right) = \Gamma\,.\,\lambda$, except in cases which will afterwards be considered, where the condition $A_{-s} = A_s$ is unnecessary.)

Hence, observing the equations (4), (9), (10),

$$\frac{\sqrt{(1-e^2)}\,f\varepsilon^{u\sqrt{(-1)}}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}(1 - e\cos u)} = \Sigma_{-\infty}^\infty \cos rm\,\underbrace{2\cos r\chi\lambda f\lambda} \ldots (13);$$

from which, assuming a system of equations analogous to (1), and representing by $\Pi(\Phi)$ the product $\Phi_1\Phi_2 \ldots$, it is easy to deduce

$$\Pi\left\{\frac{\sqrt{(1-e^2)}}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}(1 - e\cos u)}\right\}.f\{\varepsilon^{u_1\sqrt{(-1)}},\ \varepsilon^{u_2\sqrt{(-1)}}\ldots\}$$
$$= \Sigma_{-\infty}^\infty\,\Sigma_{-\infty}^\infty \ldots \Pi\cos rm\,\underbrace{\Pi(2\cos r\chi\lambda)f(\lambda_1, \lambda_2\ldots)} \ldots (14),$$

where $\Gamma\,.\,(\lambda_1, \lambda_2\ldots)$ being expansible in the form

$$\Gamma.(\lambda_1, \lambda_2\ldots) = \Sigma_{-\infty}^\infty\,\Sigma_{-\infty}^\infty \ldots A_{s_1, s_2\ldots}\lambda_1^{s_1}\lambda_2^{s_2}\ldots \quad [A_{s_1, s_2\ldots} = A_{-s_1, -s_2\ldots}]$$
$$\ldots (15).$$

$$\Gamma.(\lambda_1, \lambda_2 \ldots) = \Sigma_0^\infty\, \Sigma_0^\infty \ldots \frac{1}{2^N} A_{s_1, s_2 \ldots}\, \lambda_1^{s_1} \lambda_2^{s_2} \ldots\ldots\ldots\ldots (16),$$

N being the number of exponents which vanish.

The equations (13) and (14) may also be written in the forms

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^\infty \cos rm\; 2 \cos r\chi\lambda \frac{\sqrt{(-1)}\chi'\lambda\,\{1 - \frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}} f\lambda \ldots (17),$$

$$f\{\varepsilon^{u_1\sqrt{(-1)}}, \varepsilon^{u_2\sqrt{(-1)}} \ldots\}$$

$$= \Sigma_{-\infty}^\infty\, \Sigma_{-\infty}^\infty \ldots \Pi(\cos rm)\, \Pi \left\{2 \cos r\chi\lambda . \frac{\sqrt{(-1)}\chi'\lambda\{1 - \frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}}\right\} f(\lambda_1, \lambda_2 \ldots) \ldots (18).$$

As examples of these formulæ, we may assume

$$\chi\{\varepsilon^{u\sqrt{(-1)}}\} = m = u - e \sin u \ldots\ldots (19).$$

Hence, putting

$$\lambda^r \varepsilon^{-\frac{re}{2}(\lambda - \lambda^{-1})} + \lambda^{-r} \varepsilon^{\frac{re}{2}(\lambda - \lambda^{-1})} = \Lambda_r \ldots . (20),$$

and observing the equation

$$\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\} = 1 - e \cos u \ldots . (21),$$

the equation (17) becomes

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \Sigma_{-\infty}^\infty \cos rm\; \Lambda_r \frac{\{1 - \frac{1}{2}e(\lambda + \lambda^{-1})\}^2}{\sqrt{(1-e^2)}} \ldots (22).$$

Thus, if $$\theta - \pi = \cos^{-1} \frac{\cos u - e}{1 - e \cos u} \ldots\ldots (23),$$

assuming $$f\{\varepsilon^{u\sqrt{(-1)}}\} = \frac{\cos u - e}{1 - e \cos u} \ldots\ldots (24),$$

$$\cos(\theta - \pi) = \Sigma_{-\infty}^\infty \frac{1}{\sqrt{(1-e^2)}} \cos rm \left\{1 - \frac{e}{2}(\lambda + \lambda^{-1})\right\}\left\{\frac{1}{2}(\lambda + \lambda^{-1}) - e\right\}\Lambda_r \ldots (25),$$

the term corresponding to $r = 0$ being

$$\frac{1}{2\sqrt{(1-e^2)}}\{2\lambda - 2e - e(\lambda^2 + 1) + 2e^2\lambda\}, = -e \ldots . (26).$$

Again, assuming

$$f\{\varepsilon^{u\sqrt{(-1)}}\} = \frac{d\theta}{dm} = \frac{\sqrt{(1-e^2)}}{(1 - e \cos u)^2} \ldots . (27),$$

and integrating the resulting equation with respect to m,

$$\theta - \pi = \Sigma_{-\infty}^\infty \frac{\sin rm}{r} \Lambda_r = m + 2\Sigma_1^\infty \frac{\sin rm}{r} \Lambda_r \ldots (28),$$

a formula given in the fifth Number of the *Mathematical Journal*, and which suggested the present paper.

As another example, let

$$f\,\varepsilon^{u\sqrt{(-1)}} = \cos(\theta-\pi)\,\frac{d\theta}{dm} = \frac{\sqrt{(1-e^2)}\,.\,(\cos u - e)}{(1-e\cos u)^3} \;\ldots\ldots\; (29).$$

Then integrating with respect to m, there is a term

$$2m\,.\,\underbrace{\overbrace{\frac{\frac{1}{2}(\lambda+\lambda^{-1})-e}{1-\frac{1}{2}e(\lambda+\lambda^{-1})}}}\;\ldots\ldots\ldots\;(30),$$

which it is evident, *à priori*, must vanish. Equating it to zero, and reducing, we obtain

$$\frac{e}{1-e^2} = \underbrace{\overbrace{\frac{\lambda+\lambda^{-1}}{1-\frac{e}{2}(\lambda+\lambda^{-1})}}}\;\ldots\ldots\ldots(31),$$

i.e. $\dfrac{e}{1-e^2} = \lambda + \dfrac{e}{2}(\lambda^2+1) + \dfrac{e^2}{4}(\lambda^3+3\lambda) + \dfrac{e^3}{8}(\lambda^4+4\lambda^2+3) + \ldots\ldots(32),$

a singular formula, which may be verified by substituting for λ its value: we then obtain

$$\sin(\theta-\pi) = 2\Sigma_1^\infty\,\frac{\sin rm}{r}\,\underbrace{\overbrace{\Lambda_r\,\frac{\frac{1}{2}(\lambda+\lambda^{-1})-e}{1-\frac{1}{2}(\lambda+\lambda^{-1})}}}\;\ldots\ldots\;(33).$$

The expansions of $\sin k(\theta-\pi)$, $\cos k(\theta-\pi)$, are in like manner given by the formulæ

$$\cos k(\theta-\pi) = \Sigma_{-\infty}^{\infty}\,\underbrace{\overbrace{\Lambda_r L\cos kL}}\cos rm\;\ldots\ldots\;(34),$$

$$\sin k(\theta-\pi) = \Sigma_{-\infty}^{\infty}\,\underbrace{\overbrace{\Lambda_r\,\frac{1}{kr}\,\cos kL}}\,\frac{\sin rm}{r}\;\ldots\ldots(35),$$

where, to abbreviate, we have written

$$\cos^{-1}\left\{\frac{\frac{1}{2}(\lambda+\lambda^{-1})-e}{1-\frac{e}{2}(\lambda+\lambda^{-1})}\right\} = L\;\ldots\ldots\ldots\;(36),$$

$$\frac{\left\{1-\frac{e}{2}(\lambda+\lambda^{-1})\right\}^2}{\sqrt{(1-e^2)}} = L\;\ldots\ldots\ldots\;(37).$$

Forming the analogous expressions for

$$\cos k(\theta'-\pi'),\quad \sin k(\theta'-\pi'),$$

substituting in

$\cos k(\theta-\theta')$

$= \cos k(\pi-\pi')\,\{\cos k(\theta-\pi)\cos k(\theta'-\pi') + \sin k(\theta-\pi)\sin k(\theta'-\pi')\}$

$- \sin k(\pi-\pi')\,\{\sin k(\theta-\pi)\cos k(\theta'-\pi') - \sin k(\theta'-\pi')\cos k(\theta-\pi)\},$

and reducing the whole to multiple cosines, the final result takes the very simple form

$$\cos k(\theta-\theta')$$
$$=\Sigma_{-\infty}^{\infty}\cos\{r'm'-rm+k(\pi-\pi')\}\,\overbrace{\underbrace{\Lambda_r\Lambda'_{r'}\cos kL\cos kL'\left(L-\frac{1}{kr}\right)\left(L'-\frac{1}{kr'}\right)}}\ldots(38).$$

Again, formulæ analogous to (14), (18), may be deduced from the equation

$$\Gamma.(m_1,m_2\ldots)$$
$$=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\begin{cases}\cos(r_1m_1+r_2m_2\ldots)\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\cos(r_1m_1+r_2m_2\ldots)\,\Gamma.(m_1,m_2\ldots)+\\ \sin(r_1m_1+r_2m_2\ldots)\int_0^{2\pi}\frac{dm_1}{2\pi}\int_0^{2\pi}\frac{dm_2}{2\pi}\ldots\sin(r_1m_1+r_2m_2\ldots)\,\Gamma.(m_1,m_2\ldots)\end{cases}$$
$$\ldots\ldots(39),$$

which holds from $m_1=0$ to $m_1=2\pi$, &c., but in many cases universally. In this case, writing for $\Gamma.(m_1, m_2\ldots)$ the function

$$\Pi\left\{\frac{1}{\sqrt{(-1)}\,\chi'\{\varepsilon^{u\sqrt{(-1)}}\}}\;\frac{\sqrt{(1-e^2)}-e\sin u\sqrt{(-1)}}{1-e\cos u}\right\}f\{\varepsilon^{u_1\sqrt{(-1)}},\varepsilon^{u_2\sqrt{(-1)}}\ldots\}\ldots(40);$$

and observing

$$\frac{\sqrt{(1-e^2)}-e\sin u\sqrt{(-1)}}{1-e\cos u}=\frac{1+\lambda\varepsilon^{-u\sqrt{(-1)}}}{1-\lambda\varepsilon^{-u\sqrt{(-1)}}}$$
$$=1+2\Sigma_1^{\infty}\{\cos su-\sqrt{(-1)}\sin su\}\lambda^s\ldots\ldots(41),$$

an exactly similar analysis, (except that in the expansion $\Gamma.(\lambda_1,\lambda_2\ldots)=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots A_{s_1,s_2}\ldots\lambda_1^{s_1}\lambda_2^{s_2}\ldots$, the supposition is not made that $A_{s_1,s_2}\ldots=A_{-s_1,-s_2}\ldots$), leads to the result

$$f\{\varepsilon^{u_1\sqrt{(-1)}},\varepsilon^{u_2\sqrt{(-1)}}\ldots)\,\Pi\left\{\frac{\sqrt{(1-e^2)}-e\sin u\sqrt{(-1)}}{\sqrt{(-1)}\,\chi'\varepsilon^{u\sqrt{(-1)}}(1-e\cos u)}\right\}=$$
$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\begin{cases}\cos(r_1m_1+r_2m_2\ldots)\,\overbrace{\underbrace{2^n\cos(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1,\lambda_2\ldots)}}+\\ \sin(r_1m_1+r_2m_2\ldots)\,\overbrace{\underbrace{2^n\sin(r_1\chi_1\lambda_1+r_2\chi_2\lambda_2\ldots)f(\lambda_1,\lambda_2\ldots)}}\end{cases}$$
$$\ldots\ldots(42).$$

(n) being the number of variables $u_1, u_2\ldots$ This may also be written

$$f\{\varepsilon^{u_1\sqrt{(-1)}},\varepsilon^{u_2\sqrt{(-1)}}\ldots\}=$$
$$\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\ldots\begin{cases}\cos(r_1m_1+\ldots)\,\overbrace{\underbrace{\cos(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{(-1)}\chi'\lambda\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}-\frac{1}{2}e(\lambda-\lambda^{-1})}\right\}f(\lambda_1,\lambda_2\ldots)}}+\\ \sin(r_1m_1+\ldots)\,\overbrace{\underbrace{\sin(r_1\chi_1\lambda_1+\ldots)\,\Pi\left\{\frac{2\sqrt{(-1)}\chi'\lambda\{1-\frac{1}{2}e(\lambda+\lambda^{-1})\}}{\sqrt{(1-e^2)}-\frac{1}{2}e(\lambda-\lambda^{-1})}\right\}f(\lambda_1,\lambda_2\ldots)}}\end{cases}$$
$$\ldots\ldots(43)$$

By choosing for $f\{\varepsilon^{u_1\sqrt{(-1)}}, \varepsilon^{u_2\sqrt{(-1)}}\ldots\}$, functions expansible without sines, or without cosines, a variety of formulæ may be obtained: we may instance

$$\frac{(\lambda - \lambda^{-1}) \times \left\{1 - \frac{e}{2}(\lambda + \lambda^{-1})\right\} \Lambda_r}{\sqrt{(1-e^2)} - \frac{e}{2}(\lambda - \lambda^{-1})} = 0 \ldots\ldots (44),$$

Λ_r having the same meaning as before.

$$\text{Also,}\quad \frac{\left\{\tfrac{1}{2}(\lambda + \lambda^{-1}) - e\right\}\left\{1 - \frac{e}{2}(\lambda + \lambda^{-1})\right\} \Lambda'_r}{\sqrt{(1-e^2)} - \frac{e}{2}(\lambda - \lambda^{-1})} = 0 \ldots\ldots (45),$$

$$\text{where}\qquad \Lambda'_r = \lambda^r \varepsilon^{-\frac{re}{2}(\lambda-\lambda^{-1})} - \lambda^{-r}\varepsilon^{\frac{re}{2}(\lambda-\lambda^{-1})} \ldots\ldots\ldots (46).$$

$$\text{Again,}\quad \frac{\left\{1 - \tfrac{1}{2}e(\lambda + \lambda^{-1})\right\}(\lambda - \lambda^{-1})\,\Lambda'_r}{1 - \tfrac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda - \lambda^{-1})} + \frac{2}{r}\Lambda_r = 0 \ldots\ldots (47),$$

and

$$\frac{\left\{1 - \tfrac{1}{2}e(\lambda+\lambda^{-1})\right\}\left\{(\lambda+\lambda^{-1}) - \tfrac{1}{2}e\right\}\Lambda_r}{1 - \tfrac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda - \lambda^{-1})} = \left\{1 - \tfrac{1}{2}e(\lambda+\lambda^{-1})\right\}\left\{(\lambda+\lambda^{-1}) - \tfrac{1}{2}e\right\}\Lambda_r, \ldots\ldots (48);$$

or, what is the same thing,

$$\frac{(\lambda - \lambda^{-1})\left\{1 - \tfrac{1}{2}e(\lambda+\lambda^{-1})\right\}\left\{(\lambda+\lambda^{-1}) - \tfrac{1}{2}e\right\}\Lambda_r}{1 - \tfrac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda - \lambda^{-1})} = 0 \ldots\ldots (49);$$

or, comparing with (44),

$$\frac{(\lambda^2 - \lambda^{-2})\left\{1 - \tfrac{1}{2}e(\lambda+\lambda^{-1})\right\}\Lambda_r}{1 - \tfrac{1}{2}\frac{e}{\sqrt{(1-e^2)}}(\lambda - \lambda^{-1})} = 0 \ldots\ldots\ldots (50),$$

which are all obtained by applying the formula (43) to the expansion of $\genfrac{}{}{0pt}{}{\sin}{\cos}(\theta - \pi)$, and comparing with the equations (25), (33).

IV.—ON SOME DEFINITE INTEGRALS.*

THE following paper contains examples of the Evaluation of Definite Integrals by means of artifices depending on the following equation, the truth of which it is easy to see, viz.

$$\int_0^x f(x)\,dx = \int_0^x f(a-x)\,dx \;\ldots\ldots\; \text{(A)}.$$

1. To find the value of $\int_0^{\frac{1}{2}\pi} \log \sin \theta\, d\theta$.

Let $$u = \int_0^{\frac{1}{2}\pi} \log \sin \theta\, d\theta,$$

then $$u = \int_0^{\frac{1}{2}\pi} \log \cos \theta \,.\, d\theta, \text{ by (A)};$$

adding these, $$2u = \int_0^{\frac{1}{2}\pi} \log \sin \theta \cos \theta \,.\, d\theta$$

$$= \int_0^{\frac{1}{2}\pi} (\log \sin 2\theta + \log \tfrac{1}{2})\, d\theta;$$

but $$\int_0^{\frac{1}{2}\pi} \log \sin 2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi} \log \sin \theta' . d\theta' \text{ (by putting } 2\theta = \theta')$$

$$= \int_0^{\frac{1}{2}\pi} \log \sin \theta'\, d\theta',$$

(since the values of the sine between 0 and $\frac{1}{2}\pi$ exactly correspond to those between $\frac{1}{2}\pi$ and π,)

$$= u;$$

therefore $$2u = u + \log \tfrac{1}{2} \int_0^{\frac{1}{2}\pi} d\theta,$$

$$u = \tfrac{1}{2}\pi \log \tfrac{1}{2}.$$

2. Hence we can find $\int_0^{\pi} \theta \log \sin \theta\, d\theta$.

For $$\int_0^{\pi} \theta^2 \log \sin \theta\, d\theta = \int_0^{\pi} (\pi - \theta)^2 \log \sin \theta\, d\theta;$$

therefore $$0 = \int_0^{\pi} (\pi^2 - 2\pi\theta) \log \sin \theta\, d\theta,$$

$$\int_0^{\pi} \theta \log \sin \theta\, d\theta = \tfrac{1}{2}\pi \int_0^{\pi} \log \sin \theta\, d\theta = \tfrac{1}{2}\pi^2 \log \tfrac{1}{2}.$$

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* From a Correspondent.

3. In like manner,

$$\int_0^\pi \theta \sin^n \theta \, d\theta = \int_0^\pi (\pi - \theta) \sin^n \theta \, d\theta$$

$$= \tfrac{1}{2}\pi \int_0^\pi \sin^n \theta \, d\theta,$$

which is a known integral.

4. To find the value of $\int_0^\pi \frac{x \sin x \, dx}{1 + \cos^2 x}$.

$$\int_0^\pi \frac{x \sin x \, dx}{1 + \cos^2 x} = \int_0^\pi \frac{(\pi - x) \sin x \, dx}{1 + \cos^2 x}$$

$$= \tfrac{1}{2}\pi \int_0^\pi \frac{\sin x \, dx}{1 + \cos^2 x}$$

$$= \tfrac{1}{2}\pi \left\{\tan^{-1} 1 - \tan^{-1}(-1)\right\} = \tfrac{1}{4}\pi^2.$$

5. Suppose $f(a) = \int_0^\pi dx \log(1 - 2a \cos x + a^2)$.

Then $$f(a) = \int_0^\pi dx \log(1 + 2a \cos x + a^2);$$

adding these, it is easily seen that

$$2f(a) = \int_0^\pi dx \log(1 - 2a^2 \cos 2x + a^4)$$

$$= \tfrac{1}{2} \int_0^{2\pi} dx \log(1 - 2a^2 \cos x + a^4)$$

(by writing x instead of $2x$)

$$= \int_0^\pi dx \log(1 - 2a^2 \cos x + a^4) = f(a^2);$$

in like manner $2f(a^2) = f(a^4)$, and so on: but if a be less than 1, and n very large,

$$f(a^n) = \int_0^\pi dx \log(1) = 0,$$

therefore $f(a) = 0$;

if a be greater than 1,

$$f(a) = \int_0^\pi dx \left\{2 \log a + \log\left(1 - \frac{2}{a} \cos x + \frac{1}{a^2}\right)\right\}$$

$$= 2\pi \log a, \text{ by the former case.}$$

It is difficult to say, *a priori*, in what cases the artifice will succeed; but it is manifest that it is chiefly applicable to circular and logarithmic functions. H. G.

V.—ON THE LINEAR MOTION OF HEAT. PART I.*

THE differential equation which expresses the linear motion of heat in an infinite solid, is

$$\frac{dv}{dt} = \frac{d^2v}{dx^2},$$

where v is the temperature at the time t, of a point at the distance x from a fixed plane, which, for brevity, may be called the *zero plane*, and the conducting power is taken as unity. Its integral may be put under two forms, one containing an arbitrary function of x, and the other containing two arbitrary functions of t. I propose to deduce the latter of these solutions from the former, and to show, so far as possible, the relation which they bear to one another, with regard to the physical problem.

The first of the solutions referred to is

$$\pi^{\frac{1}{2}} v = \int_{-\infty}^{\infty} d\alpha\, \epsilon^{-\alpha^2} f(x + 2\alpha t^{\frac{1}{2}}) \text{ . . . . } (1).$$

Let ${}_0v$, v_0, represent the values of v corresponding to $t = 0$, and $x = 0$, respectively. Hence, when $t = 0$, we have

$$\pi^{\frac{1}{2}} {}_0v = \int_{-\infty}^{\infty} d\alpha\, \epsilon^{-\alpha^2} fx,$$

or, since $\int_{-\infty}^{\infty} d\alpha\, \epsilon^{-\alpha^2} = \pi^{\frac{1}{2}}$, ${}_0v = fx$.

Hence fx is the function expressing the initial distribution of heat, which therefore is, as it should be, quite arbitrary, and sufficient for determining all the succeeding distributions of the temperature. If, however, the varying temperature of any plane, as for instance the zero plane, be subject to any condition, it is obvious that the initial distribution will cease to be altogether arbitrary, as it alone is sufficient to determine the temperatures at all future times. If, however, the initial distribution be given on the positive side of the zero plane, it is clear that a certain initial distribution on the negative side will enable us to subject the variation of the temperature of the zero plane to any condition we please. By applying this principle, we can determine, in the following manner, the variable temperature of any point

* From a Correspondent.

in the cases; first, when the initial distribution on the positive side being given, the temperature of the zero plane is a given function of the time; and secondly, when the part of the solid on the negative side is removed, and the given initial distribution of temperature on the remaining part is dissipated by radiation across the zero plane, into a medium of constant or varying temperature.

1. Let the conditions be

$$_0v = \phi x \text{ when } x > 0 \ldots\ldots (2),$$
$$v_0 = \xi t \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let ψx be the distribution on the negative side, necessary to produce (3), when ϕx is the distribution on the positive side. Hence, when $x > 0$, $fx = \phi x$, and when $x < 0$, $fx = \psi x$, and therefore (1) becomes

$$\pi^{\frac{1}{2}} v = \int_{-\frac{x}{2t^{\frac{1}{2}}}}^{\infty} da\, \varepsilon^{-a^2} \phi(x + 2at^{\frac{1}{2}}) + \int_{-\infty}^{-\frac{x}{2t^{\frac{1}{2}}}} da\, \varepsilon^{-a^2} \psi(x + 2at^{\frac{1}{2}}) \ldots\ldots (4),$$

and (3) gives

$$\pi^{\frac{1}{2}} \xi t = \int_0^{\infty} da\, \varepsilon^{-a^2} \phi(2at^{\frac{1}{2}}) + \int_{-\infty}^{0} da\, \varepsilon^{-a^2} \psi(2at^{\frac{1}{2}})$$
$$= \int_0^{\infty} da\, \varepsilon^{-a^2} \{\phi(2at^{\frac{1}{2}}) + \psi(-2at^{\frac{1}{2}})\}.$$

Hence we must find a function F, such that

$$\pi^{\frac{1}{2}} \xi t = \int_0^{\infty} da\, \varepsilon^{-a^2} F(2at^{\frac{1}{2}}) \ldots\ldots\ldots (5);$$

and, when this is done, we have, for determining ψ,

$$\psi(-x) = Fx - \phi x \ldots\ldots\ldots\ldots (6).$$

To determine F, let, in the first place, ξt be a periodical function of t, and let

$$\xi t = \Sigma\left(A_i \cos \frac{2i\pi t}{p} + B_i \sin \frac{2i\pi t}{p}\right) \ldots\ldots (7).$$

Then, by taking $p = \infty$, any unperiodical function may, by Fourier's theorem, be represented in this form. Hence the problem is reduced to that of representing terms of the form $\cos \dfrac{2i\pi t}{p}$, or $\sin \dfrac{2i\pi t}{p}$, by the definite integral $\int_0^{\infty} da\, \varepsilon^{-a^2} F(2at^{\frac{1}{2}})$. To effect this, let $p = a + \sqrt{\{\pm 2mt \sqrt{(-1)}\}}$, in the first member of the equation

$$\int_{-\infty}^{\infty} dp\, \varepsilon^{-p^2} = \pi^{\frac{1}{2}}.$$

Then, dividing by $\varepsilon^{\mp 2mt\sqrt{(-1)}}$, we have

$$\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \varepsilon^{-2a\sqrt{(mt)}\,[1\pm\sqrt{(-1)}]} = \pi^{\frac{1}{2}}\, \varepsilon^{\pm 2mt\sqrt{(-1)}}.$$

Hence, by addition and subtraction,

$$\mp\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \varepsilon^{-2a\sqrt{(mt)}}\, \begin{matrix}\sin\\ \cos\end{matrix}\,\{2a\sqrt{(mt)}\} = \pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\,(2mt)\,\ldots\ldots(a),$$

the upper sign being taken along with the sines, and the lower with the cosines. Changing the sign of $\sqrt{(mt)}$, we have

$$\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \varepsilon^{2a\sqrt{(mt)}}\, \begin{matrix}\sin\\ \cos\end{matrix}\,\{2a\sqrt{(mt)}\} = \pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\,(2mt)\,\ldots\ldots(b).$$

Hence, by addition,

$$\int_{-\infty}^{\infty} da\, \varepsilon^{-a^2}\, \{\varepsilon^{2a\sqrt{(mt)}} \mp \varepsilon^{-2a\sqrt{(mt)}}\}\, \begin{matrix}\sin\\ \cos\end{matrix}\,\{2a\sqrt{(mt)}\} = 2\pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\,(2mt);$$

or, since the multiplier of da remains the same when a is changed into $-a$,

$$\int_{0}^{\infty} da\, \varepsilon^{-a^2}\, \{\varepsilon^{2a\sqrt{(mt)}} \mp \varepsilon^{-2a\sqrt{(mt)}}\}\, \begin{matrix}\sin\\ \cos\end{matrix}\,\{2a\sqrt{(mt)}\} = \pi^{\frac{1}{2}}\, \begin{matrix}\sin\\ \cos\end{matrix}\,(2mt)\ldots(c).$$

Hence, if we put $m = \dfrac{i\pi}{p}$, we have

$$\begin{aligned}\int_{0}^{\infty} da\, \varepsilon^{-a^2}\, \Sigma\Bigg\{A_i\Big(\varepsilon^{2a\sqrt{\frac{i\pi t}{p}}} + \varepsilon^{-2a\sqrt{\frac{i\pi t}{p}}}\Big)\cos\left(2a\sqrt{\frac{i\pi t}{p}}\right)&\\ + B_i\Big(\varepsilon^{2a\sqrt{\frac{i\pi t}{p}}} - \varepsilon^{-2a\sqrt{\frac{i\pi t}{p}}}\Big)\sin\left(2a\sqrt{\frac{i\pi t}{p}}\right)\Bigg\}&\\ = \pi^{\frac{1}{2}}\, \Sigma\left(A_i\cos\frac{2i\pi t}{p} + B_i\sin\frac{2i\pi t}{p}\right)&;\end{aligned}$$

and therefore we have, for the form of the function F,

$$\begin{aligned}Fx = \Sigma\Bigg\{A_i\Big(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\Big)\cos\left(x\sqrt{\frac{i\pi}{p}}\right)&\\ + B_i\Big(\varepsilon^{x\sqrt{\frac{i\pi}{p}}} - \varepsilon^{-x\sqrt{\frac{i\pi}{p}}}\Big)\sin\left(x\sqrt{\frac{i\pi}{p}}\right)\Bigg\}&\ \ldots\ldots\ (8).\end{aligned}$$

Now, to satisfy (7),

$$A_i = \frac{2}{p}\int_0^p dt'\xi t' \cos\frac{2i\pi t'}{p}, \text{ when } i > 0,$$

$$A_0 = \frac{1}{p}\int_0^p dt'\xi t',$$

$$B_i = \frac{2}{p}\int_0^p dt'\xi t' \sin\frac{2i\pi t'}{p}.$$

Hence, the expression for Fx becomes

$$pFx = \int_0^p dt'\xi t'\,(\varepsilon^0 + \varepsilon^{-0})$$

$$+ 2\sum_1^\infty \int_0^p dt'\xi t'\left[\varepsilon^{x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x - 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right.$$

$$\left. + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x + 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right];$$

or, since $(\varepsilon^z + \varepsilon^{-z}) \cos z = \left\{\varepsilon^{z\sqrt{(-1)}} + \varepsilon^{-z\sqrt{(-1)}}\right\} \cos\left\{z\sqrt{(-1)}\right\}$,

and $(\varepsilon^z - \varepsilon^{-z}) \sin z = -\left\{\varepsilon^{z\sqrt{(-1)}} - \varepsilon^{-z\sqrt{(-1)}}\right\} \sin\left\{z\sqrt{(-1)}\right\}$,

and therefore each term of the series remains the same when i is changed into $-i$,

$$pFx = \sum_{-\infty}^{\infty} \int_0^p dt'\xi t'\left[\varepsilon^{x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x - 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right.$$

$$\left. + \varepsilon^{-x\sqrt{\frac{i\pi}{p}}} \cos\left\{\sqrt{\frac{i\pi}{p}}\left(x + 2t'\sqrt{\frac{i\pi}{p}}\right)\right\}\right] \ldots . (9).$$

If ξt be not periodical, let $p = \infty$. Then, changing the limits of t' to $-\frac{1}{2}p$ and $\frac{1}{2}p$, instead of 0 and p, and putting $\frac{i\pi}{p} = m$, $\frac{\pi}{p} = dm$, we have

$$\pi Fx = \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dt'\xi t'\left[\varepsilon^{m^{\frac{1}{2}}x} \cos\left\{m^{\frac{1}{2}}(x - 2t'm^{\frac{1}{2}})\right\}\right.$$

$$\left. + \varepsilon^{-m^{\frac{1}{2}}x} \cos\left\{m^{\frac{1}{2}}(x + 2t'm^{\frac{1}{2}})\right\}\right] \ldots . (10).$$

Hence, if we determine F from (9) or (10), and ψ from (6), the solution of the problem is found by using this result in (4).

Equation (6) shows that $\psi(x)$, the initial distribution on the negative side of the zero plane, is composed of two parts, $-\phi(x)$ and $F(-x)$. The first of these, together with ϕx, on the positive side, would obviously have the effect of retaining the temperature of the zero plane at zero. But, in addition to them, there is the distribution $F(-x)$, on the negative side, which is so determined from (9) or (10), that it alone would have the effect of making the subsequent temperature of the zero plane be ξt. Hence, since the result of the two initial distributions coexisting is equal to the sum of the results in the cases in which they exist separately, it follows that, on the whole, the varying temperature of the zero plane is ξt. Hence we see how it is that, without altering the

initial distribution on the positive side, the initial temperature on the negative side may be so distributed as to make the temperature of the zero plane be ξt.

From (9) and (5), and from (10) and (5), we have the following theorems:

$$\left.\begin{aligned} p\pi^{\frac{1}{2}}\xi t = \sum_{-\infty}^{\infty}\int_0^p dt'\xi t'\int_0^\infty d\alpha\,\varepsilon^{-\alpha^2}\Bigg[\varepsilon^{2\alpha\sqrt{\frac{i\pi t}{p}}}\cos\left\{2\sqrt{\frac{i\pi}{p}}\left(\alpha t^{\frac{1}{2}} - t'\sqrt{\frac{i\pi}{p}}\right)\right\} \\ + \varepsilon^{-2\alpha\sqrt{\frac{i\pi t}{p}}}\cos\left\{2\sqrt{\frac{i\pi}{p}}\left(\alpha t^{\frac{1}{2}} + t'\sqrt{\frac{i\pi}{p}}\right)\right\}\Bigg] \\ \pi^{\frac{3}{2}}\xi t = \int_{-\infty}^{\infty} dm\int_{-\infty}^{\infty} dt'\,\xi t'\int_0^\infty d\alpha\,\varepsilon^{-\alpha^2}\big[\varepsilon^{2\alpha\sqrt{(mt)}}\cos\{2m^{\frac{1}{2}}(\alpha t^{\frac{1}{2}} - t'm^{\frac{1}{2}})\} \\ + \varepsilon^{-2\alpha\sqrt{(mt)}}\cos\{2m^{\frac{1}{2}}(\alpha t^{\frac{1}{2}} + t'm^{\frac{1}{2}})\}\big] \end{aligned}\right\} \quad (11),$$

the first or second being used according as ξt is or is not periodical.

These theorems obviously hold when t is negative as well as when it is positive. Hence we have found the distribution on the negative side of the zero plane, which not only produces in every succeeding time the given temperature of the zero plane, but would also follow if, for negative values of t, the temperature had been the same function of these negative values. In general, however, the temperature of any plane except the zero plane, as given by (4), will be impossible for negative values of t, since, except on a particular assumption with respect to ϕx, or the value of ${}_0v$, when x is positive, the initial distribution, represented by ψx and ϕx, is not of such a form as to be any stage, except the first, in a system of varying possible temperatures, or is not producible by any previous possible distribution. Thus, if ${}_0v = 0$ when x is positive, and ${}_0v = F(-x)$ when x is negative, the state represented cannot be the result of any *possible* distribution of temperature which has previously existed, though if in (4) we put $\phi x = 0$, and give t a negative value, we find a distribution, probably impossible except when $x = 0$, which will produce the distribution ${}_0v$, when $t = 0$.

VI.—ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By B. Bronwin.

In my former paper on this subject, several differential equations were integrated in finite terms, which by the ordinary methods would not have been done without an infinite series. As there is some difficulty in ascertaining the required form of the integral, I shall here exhibit a method by which that may be always done.

Let $$A_0 \frac{d^2y}{dx^2} + B_0 \frac{dy}{dx} + C_0 y = 0.$$

Differentiating and eliminating y, we obtain

$$A_1 \frac{d^3y}{dx^3} + B_1 \frac{d^2y}{dx^2} + C_1 \frac{dy}{dx} = 0.$$

$$A_1 = A_0 C_0,\ \ B_1 = B_0 C_0 + \overset{1}{A}_0 C_0 - A_0 \overset{1}{C}_0,\ \ C_1 = C_0^2 + \overset{1}{B}_0 C_0 - B_0 \overset{1}{C}_0;$$

$$\overset{1}{A}_0 = \frac{dA_0}{dx},\ \ \overset{1}{B}_0 = \frac{dB_0}{dx},\ \ \&c.$$

In like manner, repeating the same operations, we shall have

$$A_2 \frac{d^4y}{dx^4} + B_2 \frac{d^3y}{dx^3} + C_2 \frac{d^2y}{dx^2} = 0,\quad A_3 \frac{d^5y}{dx^5} + B_3 \frac{d^4y}{dx^4} + C_3 \frac{d^3y}{dx^3} = 0;$$

. . . .

$$A_r \frac{d^{r+2}y}{dx^{r+2}} + B_r \frac{d^{r+1}y}{dx^{r+1}} + C_r \frac{d^ry}{dx^r} = 0.$$

Let the following be $A_{r+1} \dfrac{d^{r+3}y}{dx^{r+3}} + B_{r+1} \dfrac{d^{r+2}y}{dx^{r+2}} = 0$; the last term having vanished.

If we eliminate $\dfrac{dy}{dx}$ between the first and second of these, and $\dfrac{d^2y}{dx^2}$ between that result and the third, and so on; we shall ultimately have $\quad y = M \dfrac{d^{r+1}y}{dx^{r+1}} + N \dfrac{d^{r+2}y}{dx^{r+2}},$

M and N being finite functions of x. Integrating the final equation of the series, we have

$$\frac{d^{r+2}y}{dx^{r+2}} = aX,\quad \frac{d^{r+1}y}{dx^{r+1}} = a\int X dx + b,$$

X a known function of x, and a and b arbitraries. Substituting these values in the expression of y, we have

$$y = a\,(M\textstyle\int Xdx + NX) + bM,$$

the complete integral.

If C_0 be constant, M will be an integer function of x; and since $y = bM$ is a particular integral, the proposed equation will give it immediately by series. The integral $\int Xdx$ must be reduced to the simplest form possible. Then, putting $M\int Xdx + NX$ for y in the proposed, we shall have N by series. This is the easiest way of obtaining M and N when C_0 is constant; but if it be a function of x, these quantities are fractions, and can only be obtained by elimination as before explained. By this method we should easily find the second particular integrals

$$y = uc^{qx} + v\int\frac{dx}{x}\,c^{qx}, \quad y = uc^{\frac{1}{2}q^2x^2} + v\int dx c^{\frac{1}{2}q^2x^2}$$

given in my former paper.

Let us now illustrate the method by one or two examples.

Suppose $\quad (a + bx)\dfrac{d^2y}{dx^2} + (f + gx)\dfrac{dy}{dx} - gy = 0\,;$

differentiate, and $\quad (a + bx)\dfrac{d^3y}{dx^3} + (b + f + gx)\dfrac{d^2y}{dx^2} = 0.$

Therefore making $\quad \beta = \dfrac{ag}{b^2} - \dfrac{f}{b} - 1, \quad \gamma = \dfrac{g}{b}$, we have

$$\frac{d^2y}{dx^2} = C\,(a + bx)^\beta\, c^{-\gamma x}, \quad \frac{dy}{dx} = C\textstyle\int dx c^{-\gamma x}(a + bx)^\beta + C \text{ and}$$

$y = C\{c^{-\gamma x}(a + bx)^{\beta+1} + (f + gx)\int dx c^{-\gamma x}(a + bx)^\beta\} + C'(f + gx)$; the complete integral. Constants multiplying the arbitraries are omitted, as it is only changing those arbitraries.

Let $\quad (a + bx)\dfrac{d^2y}{dx^2} + (f + gx)\dfrac{dy}{dx} - 2gy = 0\,;$

we have, by differentiation,

$$(a + bx)\frac{d^3y}{dx^3} + (b + f + gx)\frac{d^2y}{dx^2} - g\frac{dy}{dx} = 0\,;$$

$$(a + bx)\frac{d^4y}{dx^4} + (2b + f + gx)\frac{d^3y}{dx^3} = 0.$$

By elimination

$$2g^2y = (a + bx)(f + gx)\frac{d^3y}{dx^3} + \{g(a + bx) + (f + gx)(b + f + gx)\}\frac{d^2y}{dx^2}\,.$$

By integration

$$\frac{d^3y}{dx^3} = Cc^{-\gamma x}(a+bx)^\beta, \quad \frac{d^2y}{dx^2} = C\int dx c^{-\gamma x}(a+bx)^\beta + C',$$

where $\beta = \frac{ag}{b^2} - \frac{f}{b} - 2$, $\gamma = \frac{g}{b}$. Substituting the values of $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ in that of $2g^2y$, we have the complete integral.

To give an example or two in which C_0 is not constant; let

$$(a+bx+ex^2+fx^3)\frac{d^2y}{dx^2} + (g+hx+kx^2)\frac{dy}{dx} + (l-kx)y = 0;$$

where $k = -(h+l)\frac{l}{g}$. After one differentiation and elimination of y, $\frac{dy}{dx}$ also vanishes, and we obtain an equation of the form

$$P\frac{d^3y}{dx^3} + Q\frac{d^2y}{dx^2} = 0;$$

which gives

$$\frac{d^2y}{dx^2} = C\frac{l-kx}{a+bx+ex^2+fx^3} \cdot c^{-\int\frac{(g+hx+kx^2)dx}{a+bx+ex^2+fx^3}} = CX;$$

$$\frac{dy}{dx} = C\int Xdx + C'; \text{ and}$$

$$y = C\left\{\frac{a+bx+ex^2+fx^3}{l-kx}X + \left(\frac{g}{l}-x\right)\int Xdx\right\} + C'\left(\frac{g}{l}-x\right).$$

Let $$A_0\frac{d^2y}{dx^2} + (a-x)C_0\frac{dy}{dx} + C_0y = 0;$$

differentiating, we have

$$A_0\frac{d^3y}{dx^3} + \left\{\left(\frac{A_0}{C_0}\right)^1 + a - x\right\}C_0\frac{d^2y}{dx^2} = 0; \quad \frac{d^2y}{dx^2} = k\frac{C_0}{A_0}c^{-\int\frac{C_0}{A_0}(a-x)dx};$$

$$\frac{dy}{dx} = k\int\frac{C_0}{A_0}dxc^{-\int\frac{C_0}{A_0}(a-x)dx} + l;$$

k and l being arbitraries. Therefore

$$y = k\left\{c^{-\int\frac{C_0}{A_0}(a-x)dx} + (a-x)\int\frac{C_0}{A_0}dxc^{-\int\frac{C_0}{A_0}(a-x)dx}\right\} + l(a-x),$$

where k and l are changed into $-k$ and $-l$.

Without the last term vanishing, if we arrive at an equation which we can integrate, we shall obtain the integral of the proposed in the same manner. And it is obvious that the

method will apply to equations of the third order. If by successive differentiation we arrive at an equation, the two last terms of which vanish; or if only the last term vanish, and we can integrate the resulting equation of the second order; we shall obtain the integral of the proposed.

The equation $\dfrac{d^{r+2}y}{dx^{r+2}} = aX$ would give

$$y = a\int^{r+2} X dx^{r+2} + c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_{r+1} x^{r+1},$$

which is the more complex of the two particular integrals. It may be observed that $a\int^{r+2} X dx^{r+2}$ is the remainder of the series after the $r+2$ first terms. Suppose this particular integral developed by Taylor's or Maclaurin's Theorem, we shall have

$$y = \Sigma a_n x^n + \frac{1}{P(n)} \int_0^h \frac{d^{n+1}y}{dx^{n+1}} (h-x)^n dx,$$

h afterwards being changed into x.

Let $(1-x^2)\dfrac{d^2y}{dx^2} + mx\dfrac{dy}{dx} - ry = 0$; $\quad m = p+q-1,\ r = pq.$

Then $\qquad (1-x^2)\dfrac{d^3y}{dx^3} + m_1 x \dfrac{d^2y}{dx^2} - r_1 y = 0;$

$$m_1 = p_1 + q_1 - 1,\quad r_1 = p_1 q_1,\quad p_1 = p-1,\quad q_1 = q-1.$$

Hence, continuing the process, if p be integer, and we make $q = p + t$; we find

$$(1-x^2)\frac{d^{p+2}y}{dx^{p+2}} + (t-1)\,x\,\frac{d^{p+1}y}{dx^{p+1}} = 0,\quad \frac{d^{p+1}y}{dx^{p+1}} = C(1-x^2)^{\frac{t-1}{2}}.$$

Therefore the remainder is

$$\frac{C}{P(p)}\int (1-x^2)^{\frac{t-1}{2}} (h-x)^p dx.$$

Or, changing x into vx, and h into x, it is

$$C\frac{x^{p+1}}{P(p)}\int_0^1 (1-x^2v^2)^{\frac{t-1}{2}}(1-v)^p dv, \text{ or } C\frac{x^{p+1}}{P(p)}\int_0^1 \left\{1-x^2(1-u)^2\right\}^{\frac{t-1}{2}} u^p du.$$

This equation is integrated in my former paper; one of the series terminates in the case supposed, and the above is obviously the remainder of the other which does not terminate. The first term of the above remainder expanded is $\dfrac{Cx^{p+1}}{P(p)}$. This compared with the first term of the continuation of the series will give C. Thus, after finding the series in the

ordinary way, we shall find the remainder after the term $a_p x^p$, by a definite integral, which is easier than to integrate $\dfrac{d^{p+1}y}{dx^{p+1}} = aX$, $p+1$ times, and then to substitute the result in the proposed, in order to determine the $p+1$ arbitraries. The two methods, however, amount in reality to the same thing.

Again, let $\dfrac{d^2z}{dx^2} - q^2x\dfrac{dz}{dx} + q^2mz = 0$, m an integer. We find, as before,

$$\frac{d^{m+2}z}{dx^{m+2}} - q^2x\frac{d^{m+1}z}{dx^{m+1}} = 0, \quad \text{and} \quad \frac{d^{m+1}z}{dx^{m+1}} = Cc^{\frac{1}{2}q^2x^2}.$$

In this case, therefore, the remainder is

$$\frac{C}{P(m)}\int dx\, c^{\frac{1}{2}q^2x^2}(h-x)^m\, dx, \quad \text{or} \quad \frac{Cx^{m+1}}{P(m)}\int dv\, c^{\frac{1}{2}q^2x^2v^2}(1-v)^m.$$

This is always reducible to $w\,c^{\frac{1}{2}q^2x^2} + u\int dv\, c^{\frac{1}{2}q^2x^2v^2}$, w and u being finite integral functions of x. This is the remainder of the series which expresses that particular integral of the proposed which does not terminate, the series being carried to the term $a_m x^m$.

We may employ successive integration, integrating every term by parts. This will apply where the former method does not. Supposing that our equation may have the last term destroyed by $r+1$ integrations, let

$$A_0\frac{d^2y}{dx^2} + B_0\frac{dy}{dx} + C_0 y = 0.$$

Then $A_0\dfrac{dy}{dx} + P_0 y + \int Q_0 y\, dx + a = 0$, a an arbitrary. Make $y_1 = \int Q_0 y\, dx + a$; and by substitution the last becomes

$$A_1\frac{d^2y_1}{dx^2} + B_1\frac{dy_1}{dx} + C_1 y_1 = 0.$$

Continuing this process, we have at length

$$A_r\frac{d^2y_r}{dx^2} + B_r\frac{dy_r}{dx} + C_r y_r = 0, \qquad A_r\frac{dy_r}{dx} + P_r y_r = C,$$

the integral $\int Q_r y\, dx$ vanishing. This integrated gives

$$y_r = Cc^{-\int\frac{P_r}{A_r}dx}\int\frac{dx}{A_r}\, c^{\int\frac{P_r}{A_r}dx} + C'c^{-\int\frac{P_r}{A_r}dx}.$$

Then we have

$$y = \frac{1}{Q_0}\,\frac{dy_1}{dx}, \qquad y_1 = \frac{1}{Q_1}\,\frac{dy_2}{dx}, \quad \&c.;$$

$$\text{and} \quad y = \frac{1}{Q_0} \cdot \frac{d}{dx} \left\{ \frac{1}{Q_1} \cdot \frac{d}{dx} \left(\frac{1}{Q_2} \cdots \frac{dy_r}{dx} \right) \right\}.$$

As an example,

let $$\frac{d^2y}{dx^2} + (a + 2bx)\frac{dy}{dx} + 4by = 0.$$

After two integrations, we have

$$\frac{dy_1}{dx} + (a + 2bx)\, y_1 = C, \quad y_1 = \int y dx + a, \quad y = \frac{dy_1}{dx}.$$

By integration

$$y_1 = Cc^{-ax-bx^2} \int dx c^{ax+bx^2} + C'c^{-ax-bx^2};$$

and hence, if $M = ax + bx^2$,

$$y = C\left\{(a + 2bx)\, c^{-M} \int dx c^M - 1\right\} + C'(a + 2bx)\, c^{-M}.$$

If $\dfrac{d^2y}{dx^2} + (a + 2bx)\dfrac{dy}{dx} + 2rby = 0$; we have, obviously,

$$\frac{dy_r}{dx} + (a + 2bx)\, y_r = C, \text{ and } y_r = Cc^{-ax-bx^2} \int dx c^{ax+bx^2} + C'c^{-ax-bx^2}$$

Here we should have r differentiations to perform to find y. This might be impracticable. We should therefore make $y = zc^{-ax-bx^2}$, and we have

$$\frac{d^2z}{dx^2} - (a + 2bx)\frac{dz}{dx} + 2(r - 1)\, by = 0.$$

This will give a particular integral by descending series, viz. $z = a_{r-1}\, x^{r-1} + a_{r-2}\, x^{r-2} + \ldots\ldots + a_0$. Let this be called v, and make $z = uc^{ax+bx^2} + v \int dx c^{ax+bx^2}$. Substituting this value for z in the above equation, we shall determine u in the same manner as in several examples of my former paper.

The modes we have employed very nearly approach to certain transformations, which we will briefly notice.

Let $$x\left(\frac{d^2y}{dx^2} + r^2y\right) + q\frac{dy}{dx} = 0.$$

Make $$\frac{d^2y}{dx^2} + r^2y = y; \text{ and we have}$$

$$x\left(\frac{d^2y_1}{dx^2} + r^2y_1\right) + (q + 2)\frac{dy_1}{dx} = 0.$$

Make $x\dfrac{dy}{dx} + (q - 1)\, y = y_1$; and the same equation becomes

$$x\left(\frac{d^2y_1}{dx^2} + r^2y_1\right) + (q - 2)\frac{dy_1}{dx} = 0.$$

Thus we can increase or diminish q by any multiple of 2.

Let $$\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + ray = 0.$$

Make $\dfrac{dy}{dx} + axy = y_1$; and the transformed is

$$\frac{d^2y_1}{dx^2} + ax\frac{dy_1}{dx} + (r-1)\,ay_1 = 0,$$

where the coefficient of the last term is diminished of unity, and by repeating the process, if r be integer, may be destroyed. But we have already integrated this.

Let $$x^2\left(\frac{d^2y}{dx^2} + q^2y\right) = my;\quad m = r(r+1).$$

Make $\dfrac{dy}{dx} + \dfrac{r}{x}y = y_1$. Then we find

$$x^2\left(\frac{d^2y_1}{dx^2} + q^2y_1\right) = m_1y_1;\quad m_1 = (r-1)\,r,$$

and, consequently, r is diminished of a unit, and by a repetition of the process the last term may be destroyed.

Let $$x\left(\frac{d^2y}{dx^2} - q^2y\right) = 2rqy.$$

Make $y = zc^{\pm qx}$; and there results

$$x\frac{d^2z}{dx^2} \pm 2qx\frac{dz}{dx} = 2rqz.$$

A particular integral of each of these, $\pm r$ being integer, has been found in finite terms in the former paper. They will give the complete integral of the proposed.

If $x\dfrac{d^2y}{dx^2} + \frac{1}{2}\dfrac{dy}{dx} + by = 0$, $y = Cc^{2b^{\frac{1}{2}}x^{\frac{1}{2}}\sqrt{(-1)}} + C'c^{-2b^{\frac{1}{2}}x^{\frac{1}{2}}\sqrt{(-1)}}$.

Therefore $$x\frac{d^2y}{dx^2} + (\tfrac{1}{2} \pm i)\frac{dy}{dx} + by = 0,$$

i an integer, is always integrable in finite terms.

Many of the equations treated of in these papers seem particularly adapted for the application of a definite integral. There is, however, great difficulty in that application; and in some cases I have been unable to apply it. I will put down a few examples where it succeeds, but the form has been, for the most part, difficult to find.

Let $\dfrac{d^2y}{dx^2} + q^2x\dfrac{dy}{dx} + mq^2y = 0$. The integral is

$$y = C\int_0^\infty c^{-\frac{1}{2}t^2}t^{m-1}\,dt\,\sin(qxt) + C'\int_0^\infty c^{-\frac{1}{2}t^2}t^{m-1}\,dt\,\cos(qxt).$$

When m is integer, one of these may be integrated in finite terms.

Let $$x\frac{d^2y}{dx^2}+2q\frac{dy}{dx}+r^2xy=0,$$

q between the limits 0 and 1,

$$y=C\int_0^1(1-t^2)^{q-1}\,dt\cos(rxt)+C'x^{1-2q}\int_0^1(1-t^2)^{-q}\,dt\cos(rxt).$$

These, by expanding $\cos(rxt)$, and integrating, will give the forms obtained by series.

Mr. Hymers (see *Differential Equations*, p. 84) is mistaken in supposing that $u=\beta\int(t^2-n^2)^m\,dt\cos(xt+\alpha)$ is a complete integral.

It reduces to $A\int(t^2-n^2)^m\,dt\cos(xt)+B\int(t^2-n^2)^m\,dt\sin(xt)$. The latter of these integrals is zero, the negative and affirmative parts destroying one another.

Supposing q positive, the complete integral of

$$x\frac{d^2y}{dx^2}+2q\frac{dy}{dx}+r^2xy=0,$$

is $y=C\int_0^1 t^{q-1}dt(1-t)^{q-1}\cos(2rxt-rx)+C'\int_0^1 t^{q-1}dt(1-t)^{q-1}\sin(2rxt-rx)$;

and that of $$x\frac{d^2y}{dx^2}+2q\frac{dy}{dx}-r^2xy=0,$$

is $y=Cc^{rx}\int_0^1 t^{q-1}\,dt(1-t)^{q-1}c^{-2rxt}+C'c^{-rx}\int_0^1 t^{q-1}\,dt(1-t)^{q-1}c^{2rxt}$.

From what has been done in these papers, it appears that we may sometimes, by series, immediately obtain a particular integral in finite terms, and then by transforming the equation we may obtain the other in finite terms also; and sometimes when we can obtain neither, by transformation we shall have both in finite terms. There is one circumstance respecting certain equations which might perplex, and which therefore requires to be noticed. The equation $x^2\frac{d^2y}{dx^2}-m\frac{dy}{dx}-ry=0$, $r=p(p-1)$ (see former paper) gives immediately $y=Cv$, a series that terminates; and then, making $y=zc^{-\frac{m}{x}}$, we find $z=C'w$ in finite terms, both by descending series. But if we develop $wc^{-\frac{m}{x}}$ we find it to give v. The fact is that the series breaks, certain of the terms vanish; it then goes on again; and the continuation is really the second integral of the proposed, which is only a continuation of the series which gave the first, after certain intermediate terms have disappeared.

VII.—ON ELIMINATION.

By R. MOON, M.A. Fellow of Queens' College.

WE purpose, in the following paper, to indicate an easy practical method of obtaining the principal or symmetrical factor of the result of elimination between two functions of the same number of dimensions.

The result of the elimination of x from the quantities

$$ax + b,$$
$$\alpha x + \beta,$$
is $$a\beta - \alpha b.$$

To find the principal factor R of the result of elimination between

$$ax^2 + bx + c,$$
$$\alpha x^2 + \beta x + \gamma.$$

Suppose c and α each $= 0$, the result of elimination between

$$ax + b,$$
$$\beta x + \gamma,$$
is $$a\gamma - b\beta.$$

Multiply this by $(a\gamma)$, and we have

$$a^2\gamma^2 - ab\,\beta\gamma.$$

For $a^2\gamma^2$ put $(a\gamma - \alpha c)^2$,

$ab\,\beta\gamma$ ... $(a\beta - \alpha b)(b\gamma - \beta c)$,

and we have

$$R = (a\gamma - \alpha c)^2 - (a\beta - \alpha b)(b\gamma - \beta c).$$

Let R be the principal factor of the result of elimination between

$$ax^3 + bx^2 + cx + d,$$
$$\alpha x^3 + \beta x^2 + \gamma x + \delta;$$

the result of elimination between

$$ax^2 + bx + c,$$
$$\beta x^2 + \gamma x + \delta,$$

is $$(a\delta - \beta c)^2 - (a\gamma - \beta b)(b\delta - \gamma c),$$

or $$a^2\delta^2 + \beta^2 c^2 - 2a\delta\,\beta c$$
$$- a\gamma\, b\delta + a\gamma^2 c + b^2\beta\delta - b\beta\,\gamma c;$$

or, multiplying by $(a\delta)$,

$$a^3\delta^3 + a\delta\beta^2c^2 - 2a^2\delta^2\beta c - a^2\gamma b\delta^2$$
$$+ a^2\gamma^2 c\delta + ab^2\beta\delta^2 - ab\beta c\gamma\delta.$$

For $+a^3\delta^3$ put $+(a\delta - \alpha d)^3$

$+ab^2\beta\delta^2$ ... $+(a\beta - \alpha b)(b\delta - \beta d)^2$

$+a^2c\gamma^2\delta$ ... $+(a\gamma - \alpha c)^2(c\delta - \gamma d)$

$-a^2b\gamma\delta^2$ ... $-(a\gamma - \alpha c)(b\delta - \beta d)(a\delta - \alpha d)$

$-2a^2\beta c\delta^2$ ... $-2(a\beta - \alpha b)(c\delta - \gamma d)(a\delta - \alpha d)$

$+a\beta^2c^2\delta - ab\beta\gamma c\delta$, or

$+(a\beta c\delta)(\beta c - b\gamma)$ ... $-(a\beta - \alpha b)(c\delta - \gamma d)(b\gamma - \beta c)$;

$$\begin{aligned}\therefore\ R = (a\delta - \alpha d)^3 &+ (a\gamma - \alpha c)^2(c\delta - \gamma d)\\ &+ (a\beta - \alpha b)(b\delta - \beta d)^2\\ &- (a\gamma - \alpha c)(b\delta - \beta d)(a\delta - \alpha d)\\ &- 2(a\beta - \alpha b)(a\delta - \alpha d)(c\delta - \gamma d)\\ &- (a\beta - \alpha b)(c\delta - \gamma d)(b\gamma - \beta c).\end{aligned}$$

It may be observed, that had we treated $a\beta^2c^2\delta$ in the same way as the quantities which preceded it, we should have had either $-(a\beta - \alpha b)(b\gamma - \beta c)(c\delta - \gamma d)$

or else $(a\delta - \alpha d)(b\gamma - \beta c)^2$.

Neither of which quantities reduce themselves to $a\beta^2c^2\delta$, when $\alpha = 0$, $d = 0$; and the same may be said of $ab\beta\gamma c\delta$, whereas it will be found that each of the quantities as above substituted, *will* reduce itself to its corresponding quantity when $\alpha = 0$, $d = 0$.

By the help of this last remark, we may proceed to the case of four dimensions, of which we shall however only exhibit the result, which is, if we put $a\beta - \alpha b = f(a\ B)$ similarly for the other quantities,

$$\begin{aligned}
&+\{f(a\epsilon)\}^4 + \{f(a\gamma)\}^2\{f(c\epsilon)\}^2\\
&-\{f(a\epsilon)\}^2\{f(a\delta)f(b\epsilon) + 2f(a\gamma)f(c\epsilon) + 3f(a\beta)f(d\epsilon)\}\\
&-\{f(a\gamma)\}^2\{f(c\delta)f(d\epsilon) + 2f(b\epsilon)f(d\epsilon)\}\\
&-f(a\gamma)f(c\epsilon)\{f(a\delta)f(b\epsilon) + f(a\beta)f(d\epsilon)\}\\
&-\{f(c\epsilon)^2\}\{f(a\beta)f(b\gamma) + 2f(a\beta)f(a\delta)\}\\
&+f(a\gamma)f(a\delta)f(b\epsilon)f(d\epsilon)\\
&+f(c\epsilon)f(a\beta)f(b\delta)f(b\epsilon)\\
&-\{f(a\delta)^3\}f(d\epsilon)\\
&-\{f(b\epsilon)^3\}f(a\beta)\\
&+f(a\delta)f(b\epsilon)f(a\beta)f(d\epsilon)\\
&+f(c\delta)f(b\gamma)f(a\beta)f(d\epsilon)\\
&+2f(a\delta)f(c\delta)f(a\beta)f(d\epsilon)\\
&+2f(b\epsilon)f(b\gamma)f(a\beta)f(d\epsilon)\\
&+f(a\epsilon)f(c\epsilon)\{f(a\delta)^2 + 3f(a\beta)f(b\epsilon)\}\\
&+f(a\epsilon)f(a\gamma)\{f(b\epsilon)^2 + 3f(a\delta)f(d\epsilon)\}.
\end{aligned}$$

VIII.—EVALUATION OF CERTAIN DEFINITE INTEGRALS.

By R. L. Ellis, B.A. Fellow of Trinity College.

When the value of a definite integral is known, we may, if it involve an arbitrary parameter, integrate it (under certain conditions) with respect to this quantity. The result thus obtained involves an arbitrary constant of integration; in order to eliminate it, we may ascribe two different values to the quantity for which the integration has been effected, and then, of the two corresponding equations thus got, subtract one from the other. In other words, we integrate between limits for the arbitrary parameter, and thus get a new definite integral, involving two arbitrary quantities, namely, the two limiting values ascribed to the single one involved in the original integral. We may integrate again, with respect to either of these, and so on. But this method of proceeding, though it will lead to a variety of particular results, is not well fitted to show the nature of the class of definite integrals to which they all belong, and which may be obtained by repeated integrations for an arbitrary parameter.

If we integrate n times successively, we shall introduce n constants. These may be eliminated *at once*, in the manner I am about to point out. The result thus got, includes for every original definite integral, all that can be deduced from it by n integrations for an arbitrary parameter.

The following theorem will serve to illustrate the general method.

If Fx is a rational and integral function of circular functions of x, (sines and cosines), then we may express in finite terms the value of $\int_{-\infty}^{+\infty} \frac{Fx}{x^n}\, dx$, n being a positive integer, and such that $\left.\frac{Fx}{x^n}\right\}_0$ is not infinite.

This theorem applies to several remarkable definite integrals, some of which occur in the theory of probabilities; there are others which do not seem to have been noticed.

Dem. Fx, as every function of x, may be considered the sum of two functions, one of which remains unchanged when x changes its sign, and the other changes its sign with that of x; its value *aux signes près* remaining unaltered. Hence, whether n is odd or even, we may write

$$\frac{Fx}{x^n} = \frac{fx}{x^n} + \frac{\phi x}{x^n},$$

where $$\frac{f(-x)}{(-x)^n} = \frac{fx}{x^n}, \quad \text{and} \quad \frac{\phi(-x)}{(-x)^n} = -\frac{\phi x}{x^n}.$$

It is obvious that

$$\int_{-\infty}^{+\infty} \frac{Fx}{x^n}\, dx = 2\int_0^{\infty} \frac{fx}{x^n}\, dx;$$

and that if n is odd, fx, which is of course a rational and integral function of circular functions (sines and cosines) of x, must be developable in a series of sines exclusively, and if n is even in a series of cosines exclusively.

Thus, we may assume

$$fx = \Sigma A \begin{matrix}\sin\\ \cos\end{matrix}\Big\}\, ax \ldots\ldots\ldots\ldots (1).$$

Now, as $\dfrac{fx}{x^n}$ is not infinite when $x = 0$, the lowest power of x which can enter into fx must be not $< n$, call it m, and develope in powers of x, every sine or cosine which appears on the second side of the last written equation. We must have $\Sigma A a^{m-2} = 0$, $\Sigma A a^{m-4} = 0$, &c. ... $\frac{1}{2} m - 1$ equations if m is even, and $\frac{1}{2}(m-1)$ if it is odd.

Let us now consider the definite integral

$$\int_0^{\infty} e^{-ax} \cos rx dx = \frac{a}{a^2 + r^2}.$$

Integrating it repeatedly for r, we get

$$\int_0^{\infty} e^{-ax} \frac{\sin rx}{x}\, dx = \tan^{-1}\frac{r}{a}$$

$$\int_0^{\infty} e^{-ax} \frac{\cos rx}{x^2}\, dx = -r \tan^{-1}\frac{r}{a} + a \log\sqrt{(a^2 + r^2)} + C,$$

and generally

$$\int_0^{\infty} e^{-ax} \frac{\begin{matrix}\sin\\ \cos\end{matrix}\Big\}\, rx}{x^n}\, dx = \pm \frac{r^{n-1}}{[n-1]} \tan^{-1}\frac{r}{a} + aF_1(ra)$$
$$+ Cr^{n-2} + C_1 r^{n-4} + \text{\&c.} \ldots\ldots\ldots (2),$$

where $F_1(ra)$ does not become infinite for $a = 0$.

Replace r by every quantity represented in (1) by the general symbol a. Multiply each result by the corresponding coefficient A, and add.

Then, in virtue of the conditions,

$$\Sigma A a^{m-2} = 0, \quad \Sigma A a^{m-4} = 0, \quad \text{\&c.}$$

we shall have

$$\int_0^{\infty} e^{-ax} \frac{fx}{x^n}\, dx = \pm \Sigma \frac{\Sigma A a^{n-1}}{[n-1]} \tan^{-1}\frac{a}{a} + a\Sigma A F_1(aa).$$

Put $a = 0$, then $\tan^{-1}\dfrac{a}{a} = \pm\dfrac{\pi}{2}$, according to whether a is $>$ or $<$ than zero.

Thus we have

$$\int_0^\infty \frac{fx}{x^n}\,dx = \pm\frac{\pi}{2\,[n-1]}\,\Sigma \pm A a^{n-1} \ldots\ldots\ldots\ (3).$$

From hence, the truth of our theorem is obvious.

Of the ambiguous signs outside the symbol of summation, the upper is to be taken when n is of the forms $4p$, or $4p+1$.

When a is positive, we must take the upper of the ambiguous signs under the Σ.

It will be remarked, that in obtaining (3) we have eliminated all the constants at once, instead of getting rid of them one by one by particular conditions at each successive integration, and that the generality of this method enables us to recognize a class of definite integrals, which are all deduced from the known value of $\int_0^\infty e^{-xx}\cos rx\,dx$.

Equation (3) admits of several remarkable applications. Thus let us suppose $n=3$ and $fx = \sin ax \sin bx \sin cx$: then

$$fx = -\tfrac{1}{4}\{\sin(a+b+c)\,x - \sin(-a+b+c)\,x - \sin(a-b+c)\,x - \sin(a+b-c)\,x\}.$$

Consequently, we have by (3)

$$\int_0^\infty \frac{\sin ax \sin bx \sin cx}{x^3}\,dx = \frac{\pi}{4}\{s^2 \mp (s-a)^2 \mp (s-b)^2 \mp (s-c)^2\}$$

where, as in trigonometrical formulæ,

$$2s = a+b+c:$$

the upper sign is to be taken when the quantity to which it is affixed is > 0.

Likewise $\displaystyle\int_0^\infty \frac{\sin ax \sin bx \sin cx}{x}\,dx = \frac{\pi}{8}\,(1 \mp 1 \mp 1 \mp 1)$

where the signs follow the same rule as in the former case; the different unities involved being the zero powers of s, $s-a$, &c.

Let us now suppose that $fx = \sin^m x \cos zx$; the corresponding integral, viz. $\displaystyle\int_0^\infty \frac{\sin^m x \cos zx}{x^n}\,dx$ occurs in the theory of probabilities. Its value is given at p. 170 of *Laplace's Théorie des Probabilities*, where it is obtained by a method founded on a transition from real to imaginary quantities. The nature of what are called imaginary quantities is certainly better understood than it was some time since; but it seems to have been the opinion of Poisson, as well as of Laplace himself, that results thus obtained require confirmation. In this view I confess I do not acquiesce; but if only in deference

to their authority, it may be desirable to show how readily imaginary quantities may be avoided in estimating the value of the integral in question.

$$\sin^m x = \pm\frac{1}{2^{m-1}}\left\{\cos mx - \frac{m}{1}\cos(m-2)x + \&c.\right\} \text{ if } m \text{ is odd,}$$

$$\text{and } = \pm\frac{1}{2^{m-1}}\left\{\sin mx - \frac{m}{1}\sin(m-2)x + \&c.\right\} \text{ if it is even.}$$

Hence, by (3), $\displaystyle\int_0^\infty \frac{\sin^m x \cos zx}{x^n}\,dx =$

$$\frac{\pi}{[n-1]\,2^m}\left\{(m+z)^{n-1} \pm (m-z)^{n-1} - \frac{m}{1}\{(m+z-2)^{n-1} \pm (m-z-2)^{n-1}\} + \&c.\right\}$$

Let us suppose $z > m$; then the lower of each pair of ambiguous signs must be taken, and the expression within the brackets may be written thus

$$(m+z)^{n-1} - \frac{m}{1}(m+z-2)^{n-1} + \&c. + \frac{m}{1}(m-z-2)^{n-1} - (m-z)^{n-1} \;\ldots\ldots\; (q).$$

As, from the nature of the case, m and n are either both odd or both even, if m is even, $n-1$ is odd, and therefore $(m-z)^{n-1} = -(z-m)^{n-1}$; and thus, in every case, (q) equals

$$(1-D^{-1})^m (z+m)^{n-1} \;\ldots\ldots\; \{D\phi z = \phi(z+2) \quad \text{say,}\}$$

and this is $\quad \Delta^m D^{-m}(z+m)^{n-1} = \Delta^m (z-m)^{n-1} = 0,$

since m is $> n-1$. Consequently

$$\int_0^\infty \frac{\sin^m x}{x^n}\cos zx\,dx = 0, \text{ when } z \text{ is } > m,$$

a remark not made by Laplace; when $m = n = 1$, its truth is known.

As $(q) = 0$, add it, multiplied by $\dfrac{\pi}{[n-1]\,2^{m-1}}$, to the value of the integral already found,

$$\therefore \int_0^\infty \frac{\sin^m x}{x^n}\cos zx\,dx = \frac{\pi}{[n-1]\,2^m}\left\{(m+z)^{n-1} - \frac{m}{1}(m+z-2)^{n-1} + \&c.\right\} \ldots (4),$$

where the series stops whenever the next term would introduce a negative quantity raised to the power $n-1$. This is easily seen to be true, for every such term will have a different sign in (q), and in the definite integral, and thus on addition, all such terms will disappear. Equation (4) is Laplace's form; the discontinuity of the function is now expressed, not by ambiguous signs but by the stopping short of the series at different points.

By a similar method, we find that

$$\int_0^\infty \frac{\sin^m x}{x^{n+1}} \sin zx dx = \frac{\pi}{[n] 2^{m-1}} \left\{ (m+z)^{n-1} - \frac{m}{1}(m+z-2)^{n-1} + \&c. \right\} . . (5),$$

which might have been deduced from (4) by integrating both sides without introducing any complementary quantities. This remark is general; having once established the general form of (4) for any given value of n, we may deduce from it that which corresponds to any other value of n, simply by differentiation or by integration, without bringing in any constants. I conceive that this remark is general, and if so we may differentiate on both sides with fractional indices. Let the index be $-p$, then as

$$d^p \cos zx = x^p \cos \left(zx + p \frac{\pi}{2} \right) dz^p,$$

the first side of (4) will become

$$\int_0^\infty \frac{\sin^m x}{x^n} x^{-p} \cos \left(zx - p \frac{\pi}{2} \right) dx;$$

and the second will be

$$\frac{\pi}{[n-1+p] 2^m} \{ (m+z)^{n-1+p} - \&c. \}$$

Thus, we get

$$\int_0^\infty \frac{\sin^m x}{x^n} x^{-p} \cos \left(zx - p \frac{\pi}{2} \right) dx =$$

$$\frac{\pi}{[n-1+p] 2^m} \left\{ (m+z)^{n-1+p} - \frac{m}{1}(m+z-2)^{n-1+p} + \&c. \right\} . . . . (6).$$

If we take $n = m$ this is equivalent to Laplace's general formula (p), at p. 168 of the Théorie.

The method of this paper leads to some elegant results when applied to the definite integral $\int_0^\infty e^{-ax^2} dx$, but it is enough to point out this application, which involves no difficulty whatever.

IX.—PROPOSITIONS IN THE THEORY OF ATTRACTION.

LET x, y, z, be the co-ordinates of any point P, in an attracting or repelling body M; let dm be an element of the mass, at the point P, which will be positive or negative according as it is attractive or repulsive; let x', y', z' be the co-ordinates of an attracted point P';

let $\Delta = \{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}^{\frac{1}{2}}$;

and let
$$v' = \int \frac{dm}{\Delta},$$
the integral including the whole of M. This expression has been called by Green, the potential of the body M, on the point P', and the same name has been employed by Gauss, (in a Mémoire on "General Theorems relating to Attractive and Repulsive Forces," in the Resultate aus den Beobachtungen des magnetischen Vereins im Jahre 1839, Leipsic 1840, edited by M. Gauss and Weber.*) By a known theorem, the components of the attraction of M on P', in the directions of x, y, z, are
$$-\frac{dv'}{dx'}, -\frac{dv'}{dy'}, -\frac{dv'}{dz'},$$
and, if $d\gamma'$ be the element of any line, straight or curved, which passes through P', the attraction in the direction of this element is $\frac{dv'}{d\gamma'}$. Hence it follows that, if a surface be drawn through any point P' for every point of which the potential has the same value, the attraction on every point in the surface is wholly in the direction of the normal. Surfaces for which the potential is constant, are therefore called, by Gauss, *surfaces* of *equilibrium.* It has been shown in a former paper,† that, if M, instead of an attractive mass, were a group of sources of heat or cold, in the interior of an infinite homogeneous solid, v' would be the permanent temperature produced by them, at P'. In that case, the surfaces of equilibrium would be *isothermal surfaces.*

When the attraction of (positive or negative) matter, as for instance electricity, spread over a surface is considered, the density of the matter at any point is measured by the quantity of matter on an element of the surface, divided by that element.

The principal object of this paper is to prove the following theorems.

If upon E, one of the surfaces of equilibrium enclosing an attracting mass, its matter be distributed in such a manner that its density at any point P is equal to the attraction of M on P; then,

1. The attraction of the matter spread over E, on an external point, is equal to the attraction of M on the same point multiplied by 4π.

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* Translations of this paper have been published in Taylor's Scientific Memoirs for April 1842, and in the Nos. of Liouville's Journal for July and August 1842.

† See vol. III. p. 73 of this Journal.

2. The attraction of the matter on E, on an internal point, is nothing.

These theorems were proved in a previous paper, (see vol. III. p. 75,) from considerations relative to the uniform motion of heat; but in the following they are proved by direct integration.

Let u be the potential of M, on the point P, (xyz) in E. The components of the attraction of M on P, in the directions of x, y, z, are

$$-\frac{du}{dx},\ -\frac{du}{dy},\ -\frac{du}{dz};$$

and hence, if α, β, γ be the angles which a normal to E at P makes with these directions, the total attraction on P is

$$-\left(\frac{du}{dx}\cos\alpha + \frac{du}{dy}\cos\beta + \frac{du}{dz}\cos\gamma\right) \text{ or } -\frac{du}{dn},$$

if dn be an element of the normal through P.

This is therefore the expression for the density at P, of the matter we have supposed to be spread over E. Let ds be an element of E at P; let v' be the potential of E, on a point P', $(x'y'z')$, either within or without E; and let Δ be the distance from P to P'. Then

$$v' = -\left\{\iint \frac{\left(\frac{du}{dx}\cos\alpha + \frac{du}{dy}\cos\beta + \frac{du}{dz}\cos\gamma\right)ds}{\Delta}\right\} = -\left\{\iint \frac{\frac{du}{dn}\,ds}{\Delta}\right\} \ldots (a),$$

the brackets enclosing the integrals denoting that the integrations are to be extended over the whole surface E. Now for ds, we may choose any one of the expressions,

$$ds = \frac{dy\,dz}{\cos\alpha},\quad ds = \frac{dx\,dz}{\cos\beta},\quad ds = \frac{dx\,dy}{\cos\gamma}.$$

Hence any integral of the form

$$\{\int(A\cos\alpha + B\cos\beta + C\cos\gamma)\,ds\}$$

may be transformed into the sum of the three integrals,

$$(\iint A\,dy\,dz),\quad (\iint B\,dx\,dz),\quad (\iint C\,dx\,dy),$$

by using the first, second, and third of the expressions for ds, in the first, second, and third terms of the integral respectively.

Hence, if $\quad A = \frac{d\phi}{dx}\psi,\quad B = \frac{d\phi}{dy}\psi,\quad C = \frac{d\phi}{dz}\psi,$

$$\left(\int\frac{d\phi}{dn}\psi ds\right) \text{ or } \left\{\iint\left(\frac{d\phi}{dx}\cos\alpha + \frac{d\phi}{dy}\cos\beta + \frac{d\phi}{dz}\cos\gamma\right)\psi ds\right\},$$

$$= \left\{\iint\psi\left(\frac{d\phi}{dx}dy\,dz + \frac{d\phi}{dy}dx\,dz + \frac{d\phi}{dz}dx\,dy\right)\right\} \ldots (b),$$

the limits of the integrations relative to y and z, x and z, x and y, being so chosen as to include the whole of the surface considered.

Making use of this transformation in (a) we have

$$v' = -\left\{\iint\left(\frac{du}{dx}\,\frac{dydz}{\Delta}+\frac{du}{dy}\,\frac{dxdz}{\Delta}+\frac{du}{dz}\,\frac{dxdy}{\Delta}\right)\right\}\ldots.(a').$$

$$\text{Now}\ \iint\frac{du}{dx}\,\frac{dydz}{\Delta} = \int\int dydz\int dx\left(\frac{d^2u}{dx^2}\,\frac{1}{\Delta}+\frac{du}{dx}\,\frac{d}{dx}\,\frac{1}{\Delta}\right)$$

$$= \int\int\int dxdydz\left(\frac{d^2u}{dx^2}\,\frac{1}{\Delta}+\frac{du}{dx}\,\frac{d}{dx}\,\frac{1}{\Delta}\right).$$

Hence, if the integrals in the second member include every point in the space contained between E, and another surface of equilibrium, E', without E, and which we shall suppose to be also without P', we have

$$\left\{\iint\frac{du}{dx}\frac{dydz}{\Delta}\right\}' - \left\{\iint\frac{du}{dx}\frac{dydz}{\Delta}\right\} = \iiint\left(\frac{d^2u}{dx^2}\frac{1}{\Delta}+\frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\right)dxdydz,$$

the accent denoting that, in the term accented, the integrals are to be extended over the surface E'. Modifying in a similar manner the second and third terms of v', we have

$$\left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\}' - \left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\}\ \text{or}\ \left\{\int\frac{\frac{du}{dn}\,ds}{\Delta}\right\}' + v'$$

$$= \iiint\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+\frac{du}{dx}\frac{d}{dx}\cdot\frac{1}{\Delta}+\frac{du}{dy}\frac{d}{dy}\cdot\frac{1}{\Delta}+\frac{du}{dz}\frac{d}{dz}\cdot\frac{1}{\Delta}\right)dxdydz \quad \ldots.(c).$$

Now, for all points without M,

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

by a known theorem; and such points only are included in the integrals in the second member of (c).

Also, by integration by parts,

$$\iiint\frac{du}{dx}\frac{d}{dx}\frac{1}{\Delta}\,dxdydz = \iint u\,\frac{d}{dx}\frac{1}{\Delta}\,dydz - \iiint u\,\frac{d^2}{dx^2}\frac{1}{\Delta}\,dxdydz$$

$$= \left\{\iint u\,\frac{d}{dx}\frac{1}{\Delta}\,dydz\right\}' - \left\{\iint u\,\frac{d}{dx}\,\frac{1}{\Delta}\,dydz\right\} - \iiint u\,\frac{d^2}{dx^2}\,\frac{1}{\Delta}\,dxdydz.$$

Modifying similarly the two remaining terms of the second member of (c), we have

$$\left\{\int \frac{\frac{du}{dn}ds}{\Delta}\right\}' + v' = \left\{\iint u\left(\frac{d}{dx}\frac{1}{\Delta}dydz + \frac{d}{dy}\frac{1}{\Delta}dxdz + \frac{d}{dz}\frac{1}{\Delta}dxdy\right)\right\}'$$
$$-\left\{\iint u\left(\frac{d}{dx}\frac{1}{\Delta}\,dydz + \frac{d}{dy}\frac{1}{\Delta}\,dxdz + \frac{d}{dz}\frac{1}{\Delta}\,dxdy\right)\right\}$$
$$-\iiint u\left(\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta}\right)dxdydz \ldots . (c').$$

Now, since E and E' are surfaces of equilibrium, u is constant for each. Again,

$$\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta} = 0,$$

except when P coincides with P', at which point u has the value u'. Hence, the value of the integrals,

$$\iiint u\left(\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta}\right)dxdydz$$

is only affected by these elements, for which $u = u'$, and hence u may be taken without the integral sign, as being constant and equal to u'. If therefore, for brevity, we put

$$\iint\left(\frac{d}{dx}\frac{1}{\Delta}\,dydz + \frac{d}{dy}\frac{1}{\Delta}\,dxdz + \frac{d}{dz}\frac{1}{\Delta}\,dxdy\right) = (h) \text{ or } (h)' \ldots . (d),$$

according as the integrals refer to E, or to E',

$$\text{and } \iiint\left(\frac{d^2}{dx^2}\frac{1}{\Delta} + \frac{d^2}{dy^2}\frac{1}{\Delta} + \frac{d^2}{dz^2}\frac{1}{\Delta}\right)' dxdydz = k. \ldots \ldots . (e),$$

the integrations including every point between E and E'; equation (c') becomes

$$\left\{\int \frac{\frac{du}{dn}ds}{\Delta}\right\}' + v' = (u)'\,(h)' - (u)\,(h) - u'k \;\ldots\ldots\; (c'').$$

Now it is obvious that, at a great distance from M, the surfaces of equilibrium are very nearly spherical. Let E' be taken so far off that it may be considered as spherical, without sensible error, and let γ be the distance of any point in E', from the centre, a fixed point in M, or, which is the same, the radius of the sphere. Then $-\dfrac{du}{dn}$, or $-\dfrac{du}{d\gamma}$ is the attraction of M, on a point in E', and is therefore equal to $\dfrac{M}{\gamma^2}$,

and therefore, by the known expression for the potential of a uniform spherical shell, on an interior point,

$$-\left\{\int \frac{\frac{du}{dn}\,ds}{\Delta}\right\}', \text{ or } \frac{M}{\gamma^2}\left\{\int\frac{ds}{\Delta}\right\}' = \frac{M}{\gamma^2}\,4\pi\gamma = 4\pi\,(u)' \ldots . (f).$$

It now only remains to determine the integrals (h), $(h)'$, and k.

By putting, in (b), $\psi = 1$, $\phi = \frac{1}{\Delta}$, we find the following transformation, for (h),

$$h = \int \frac{d\frac{1}{\Delta}}{du}\,ds = -\int\frac{d\Delta}{du}\,\frac{ds}{\Delta^2}.$$

Now let the point (xyz) be referred to the polar co-ordinates, γ, θ, ϕ. Then, if P' be pole, $\gamma = \Delta$. Also, if ψ be the angle between Δ and dn, the expression for ds is

$$ds = \frac{\Delta^2 \sin\theta d\theta d\phi}{\cos\psi}, \quad \text{or, since } \cos\psi = \frac{d\Delta}{dn},$$

$$ds = \frac{\Delta^2 \sin\theta d\theta d\phi}{\frac{d\Delta}{dn}}.$$

Hence, $\qquad h = -\iint \sin\theta d\theta d\phi.$

If P' be within the surface to which the integrals refer, the limits for θ are 0 and π, and for ϕ, 0 and 2π, and in that case, $h = -4\pi$; therefore, since P' is always within E',

$$(h)' = -4\pi \ldots\ldots\ldots (g).$$

If P' be without the surface considered, then, for each value of θ, we must take the sum of the expressions

$$-\sin\theta d\theta d\phi, \text{ and } -\sin\theta\,(-d\theta)\,d\phi,$$

and therefore, each element of the integral is destroyed by another equal to it, but with a contrary sign, and the value of the complete integral is therefore zero.

Hence, according as P' is without or within E,

$$(h) = 0, \text{ or } (h) = -4\pi \ldots\ldots\ldots (h).$$

Again, to find the value of k, we have, by dividing it into three terms, and integrating each once,

$$k = \left\{\iint\left(\frac{d}{dx}\,\frac{1}{\Delta}\,dy\,dz + \frac{d}{dy}\,\frac{1}{\Delta}\,dx\,dz + \frac{d}{dz}\,\frac{1}{\Delta}\,dx\,dy\right)\right\}'$$

$$-\left\{\iint\left(\frac{d}{dx}\,\frac{1}{\Delta}\,dy\,dz + \frac{d}{dy}\,\frac{1}{\Delta}\,dx\,dz + \frac{d}{dz}\,\frac{1}{\Delta}\,dx\,dy\right)\right\}$$

$$= (h)' - (h) = -4\pi - 0, \quad \text{or} = -4\pi + 4\pi;$$

and therefore, according as P' is without or within E,

$$k = -4\pi, \quad \text{or} \quad k = 0 \; \ldots\ldots\ldots \; (k).$$

Hence, making use of (f), (g), (h), (k), in (c'), we have

$$v' = 4\pi u', \quad \text{when } P' \text{ is without } E. \ldots\ldots(1),$$

$$v' = 4\pi(u), \quad \text{when } P' \text{ is within } E. \ldots\ldots(2).$$

From the first of these equations it follows, that the attraction of E, on a point without it, is the same as that of M, multiplied by 4π; and since the second shows that the potential of E, on internal points, is constant, we infer that the attraction of E on internal points is nothing.

These theorems, along with some others which were also proved in the previous paper in this Journal, already referred to, had, I have since found, been given previously by Gauss. One of the most important of these is the following. If a mass M be wholly within, or wholly without a surface, an equal mass may be distributed over this surface in such a manner that its attraction, in the former case on external points, and in the latter on internal, will be equal to the attraction of M, on the same points. This theorem, which was proved from physical considerations in the paper *On the Uniform Motion of Heat, &c.*, is proved analytically in Gauss' *Mémoire*, but the same method is used in both to infer from it the truth of propositions (1) and (2).

From Prop. (2) it follows that, if E be the surface of an electrified conducting body, the intensity of the electricity at any point will be proportional to the attraction of M on the point. Hence we have the means of finding an infinite number of forms for conducting bodies, on which the distribution of electricity can be determined.

Thus, if M consists of a group of material points, m_1, m_2, &c., whose co-ordinates are x_1, y_1, z_1; x_2, y_2, z_2, &c., the general equation to the surfaces of equilibrium is

$$\frac{m_1}{\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\}^{\frac{1}{2}}} + \frac{m_2}{\{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2\}^{\frac{1}{2}}} + \&\text{c.} = \lambda,$$

and the intensity of electricity at any point of a solid body, bounded by one of them, will be the value of

$$\left\{\left(\frac{d\lambda}{dx}\right)^2 + \left(\frac{d\lambda}{dy}\right)^2 + \left(\frac{d\lambda}{dz}\right)^2\right\}^{\frac{1}{2}},$$

at the point.

To take a simple case: Let there be only two material points, of equal intensity. The surface will then be a surface of revolution, and will be symmetrical with regard to a plane

perpendicular, through its point of bisection, to the line joining the two points, and would probably very easily be constructed, in practice. We should thus have a simple method of verifying numerically the mathematical theory of electricity. P. Q. R.

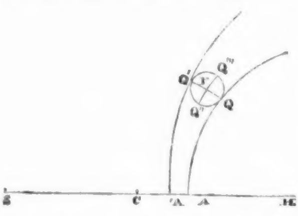
(*To be continued.*)

X.—NEW PROPERTY OF THE ELLIPSE AND HYPERBOLA.*

If a body, setting out from a given point, move so that the *difference* of its distances from two fixed points is always *greater*, or always *less*, than if it had moved over an equal space in any other way, its path will be an *ellipse*, of which the two fixed points are the foci. If it move so that the *sum* of its distances from the fixed points has that property, its path will be an *hyperbola*, of which the fixed points are the foci.

Taking the first case, let S, H, be the fixed points: then, since the difference of the distances is *always* to be the greatest or least, it must be true for every point of the path; and consequently the body, in moving from a position P over an elementary space PQ of given length, must move in such a direction as to satisfy the condition $SQ - HQ$, a maximum or a minimum.

Now, with the centre P and radius PQ, describe a circle; then, in what direction soever the body move, it must be found somewhere in the circumference of this circle when it has moved over a space equal to PQ: we have therefore to enquire, what point Q in the circumference satisfies the condition



$$SQ - HQ,$$

a maximum or a minimum. With foci S, H, describe two hyperbolas: one of which, AQ, touches the circle on the side towards H; and the other, $A'Q'$, on the side towards S. Let Q, Q', be the points of contact; and bisect SH in C. Then an hyperbola, having the foci S, H, and a major axis greater than $2CA$, would fall within AQ, and therefore would not reach the circle; AQ is therefore the hyperbola of greatest major axis which can have

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* From a Correspondent.

a point in common with the circle: consequently, $SQ - HQ$ is greater than if Q were situated in any other point of the circle; and therefore Q is the point to which the body must move, in order that the difference of its distances from S and H may always be greater than if it had moved through the same space in another direction. But PQ is a normal to the circle, and therefore to the hyperbola; and consequently SQ, HQ, are inclined at equal angles to PQ, which is an element of the required path. Hence the body moves in a curve, the tangent at every point P of which is inclined at equal angles to SP, HP; the curve is therefore an ellipse, of which S, H, are the foci. By the aid of the hyperbola $A'Q'$, we may shew that the body must move in the same ellipse to Q', if the difference of its distances from S and H is always to be the *least* possible. Hence it appears, that when a body moves in an ellipse, the difference of its focal distances is always a maximum or always a minimum, according as it is approaching towards or receding from the major axis.

If with the foci S and H we had described ellipses touching the circle in Q'' and Q''', we might have shewn that the body must move from P to Q'', or from P to Q''', in order that the *sum* of its distances from S and H may always be the least or the greatest possible. Hence $Q''PQ'''$ is ultimately an arc of an hyperbola, of which S and H are the foci. And hence, when a body moves in an hyperbola, the sum of its focal distances is always a maximum or always a minimum, according as it is receding from or approaching towards the major axis.

V.

XI.—MATHEMATICAL NOTES.

1. *Note on Mr. Bronwin's Paper on Elliptic Integrals.*—Jacobi's formulæ (8), (13), in p. 38, and the second formula in p. 37, of the *Nova Fundamenta, &c.* which Mr. Bronwin objects to in the case of m even, are perfectly correct. His own do certainly fail in that case, and the reason is obvious enough. The formulæ of Jacobi in question, adapted to the notation of the paper referred to, are

$$s.a.v = \frac{s.a.u\; s.a\,(u+2\omega)\ldots s.a\,\{u+2\,(n-1)\,\omega\}}{s.a(K-2\omega)s.a(K-4\omega)\ldots s.a\{K-2(n-1)\omega\}}\ldots.(1),$$

$$c.a.v = \frac{c.a.u.c.a\;(u+2\omega)\;\ldots\;c.a\;\{u+2\;(n-1)\;\omega\}}{c.a.2\omega.c.a.4\omega\;\ldots\;c.a\;(2n-2)\;\omega}\;\ldots\ldots(2),$$

$$\frac{1}{\beta} = \frac{s.a\,(K-2\omega)\,s.a\,(K-4\omega)\ldots s.a\,\{K-2\,(n-1)\,\omega\}}{s.a.2\omega\; s.a.4\omega\;\ldots\; s.a.2\,(n-1)\,\omega}\;\ldots(3).$$

The second of which coincides with Mr. Bronwin's, while he has for the denominator of the first and third,

$$s\,a.\omega\ s.a.3\omega \ldots\ldots s.a\,(2n-1)\,\omega,$$

a quantity which, as he remarks, vanishes when m is even, (and m'; however the passage refers particularly to $m'=0$). Jacobi's denominator does not vanish on the same supposition. Assume that from the equation (2) we may deduce one of the form m (1), only having a constant C for its denominator. In the equation (2), let u have the value ω assigned to it. $c.a.v$ contains the factor $c.a.n\omega$, which for

$$\omega = \frac{(2r+1)K + 2r'K'\sqrt{(-1)}}{n}$$

vanishes, while for

$$\omega = \frac{2rK + 2r'K'\sqrt{(-1)}}{n}, \quad \omega = \frac{2rK + (2r'+1)K'\sqrt{(-1)}}{n'},$$

$$\omega = \frac{(2r+1)K + (2r'+1)K'\sqrt{(-1)}}{n},$$

it reduces itself to $(-1)^{r+r'}$, ∞, $\dfrac{K'\sqrt{(-1)}}{n}$, respectively. In the first case the corresponding value of $s.a.v$ is of course ± 1, and we have therefore the equation

$$\pm C = s.a.\omega\ s.a.3\omega \ldots\ldots s.a\,(2n-1)\,\omega,$$

Mr. Bronwin's divisor, which is therefore equal to Jacobi's in this particular case only. The two next cases give no results, and the last gives

$$\pm \frac{1}{k}\,C = s.a.\omega.s.a.3\omega \ldots\ldots s.a\,(2n-1)\,\omega;$$

or in this case the divisor is

$$k\,.\,s.a.\omega.s.a.3\omega \ldots\ldots s.a\,(2n-1)\,\omega.$$

C.

2. *Solution of a Geometrical Problem.*—The sum of the squares of the perpendiculars let fall from n given points on a plane is constant; the plane envelopes a central surface of the second order, having its centre at the centre of gravity of the n points, and its axes coincident with the principal axes of the system of n points.

Assuming any rectangular axes, let $(x'y'z')$, $(x''y''z'')$ ...... $(x^{\text{v}}y^{\text{v}}z^{\text{v}})$, be the projective co-ordinates of the n points, ξ, v, ζ, the tangential co-ordinates of the plane, the sum of the squares $= nk^2$; then we shall have

$$P_{\prime}^2 + P_{\prime\prime}^2 + P_{\prime\prime\prime}^2 \ldots\ldots P_n^2 = nk^2,$$

or as $P_{,} = \dfrac{(x'\xi + y'\upsilon + z'\zeta - 1)}{\sqrt{(\xi^2 + \upsilon^2 + \zeta^2)}}$, $P_{,,} = \dfrac{(x''\xi + y''\upsilon + z''\zeta - 1)}{\sqrt{(\xi^2 + \upsilon^2 + \zeta^2)}}$, &c.

Squaring and adding,

$$\{x'^2 + x''^2 + x'''^2 ...\}\,\xi^2 + \{y'^2 + y''^2 + y'''^2 ...\}\,\upsilon^2 + \{z'^2 + z''^2 + z'''^2 ...\}\,\zeta^2$$
$$+ 2\,\{x'y' + x''y'' + x'''y'''\}\,\xi\upsilon + 2\,\{x'z' + x''z'' + x'''z'''\}\,\xi\zeta$$
$$+ 2\,\{y'z' + y''z'' + y'''z''' ...\}\,\upsilon\zeta + 2\,\{x'z' + x''z'' + x'''z''' ...\}\,\xi\zeta$$
$$- 2\,\{x + x' + x'' ...\}\,\xi - 2\,\{y' + y'' + y''' ...\}\,\upsilon - 2\,\{z' + z'' + z''' + z^{iv}\}\,\zeta + n$$
$$= nk^2\,\{\xi^2 + \upsilon^2 + \zeta^2\};$$

which is the tangential equation of a central surface of the second order (when the absolute term is unity in an equation of this nature, the semi-coefficients of the linear terms ξ, υ, ζ, are co-ordinates of the centre); hence the co-ordinates of the centre of the surface are

$$\frac{x' + x'' + x''' ...}{n}, \quad \frac{y' + y'' + y''' ...}{n}, \quad \frac{z' + z'' + z''' ...}{n};$$

but these are the co-ordinates of the centre of gravity of the n points: hence, let the origin of co-ordinates be supposed to have been originally placed at the centre of gravity of the points, and the axes of co-ordinates coinciding with the principal axes of the system of n points; and let a, b, c, denote the radii of gyration of the system round the axes of X, Y, Z, respectively.

Then we shall have the following equations:

$$x'^2 + x''^2 + x'''^2 ... = na^2,\; y'^2 + y''^2 + y'''^2 ... = nb^2,\; z'^2 + z''^2 + z'''^2 .. = nc^2,$$
$$x'y' + x''y'' + x'''y''' = 0,\; x'z' + x''z'' + x'''z''' ... = 0,\; y'z' + y''z'' + y'''z''' = 0,$$
$$x' + x'' + x''' ... = 0,\;\; y' + y'' + y''' + y^{iv} = 0,\;\; z' + z'' + z''' + z^{iv} = 0:$$

making these substitutions in the original equation, and dividing by n, we shall have

$$(k^2 - a^2)\,\xi^2 + (k^2 - b^2)\,\upsilon^2 + (k^2 - c^2)\,\xi^2 = 1,$$

the tangential equation of a central surface of the second order, the squares of whose semiaxes are $(k^2 - a^2)$, $(k^2 - b^2)$, and $(k^2 - c^2)$.

The distances of the foci of the principal sections of this surface from the centre are independent of k; hence, if different groups of perpendiculars are let fall from the same n points on so many different planes, these planes will envelope as many confocal surfaces of the second order.

J. B.

3. *Note on the Measure of Intensity in the Theory of Light.*—The reason assigned by Mr. Airy (*Tracts*, p. 296, note,) for taking the square of the coefficient of the disturbance as the measure of the intensity of light, appears to be not very satisfactory: the following considerations may perhaps place the matter in a clearer light. They are taken from a paper by Abria in *Liouville's Journal*, tom. IV. p. 248.

The mechanical effect of a body in motion is measured by the vis viva. Now, in order that light may produce a sensible impression on the retina, it is necessary that the action should continue for a time which, though short, is yet very much greater than the time of vibration of a particle of the luminiferous ether: therefore the measure of the effect must be the aggregate vis viva during the time necessary to produce sensation. But it is generally required only to find the ratio of the intensities at two different points; and hence it is sufficient to calculate the ratio between the sums of the squares of the velocities of the molecules at the two points during the time of one vibration: for in consequence of the great number of vibrations which take place before a sensible effect is produced, we may suppose that number to be an integer, and the same for the two points.

Now, let the disturbance be represented by

$$a \sin \frac{2\pi}{\lambda}(vt - x);$$

then the velocity of the molecule will be

$$\frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x);$$

and if τ be the time of a vibration, the sum of the vis viva during that time will be represented by

$$\int_0^\tau dt \frac{4\pi^2 v^2}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda}(vt - x).$$

But $\tau = \dfrac{\lambda}{v}$; therefore, integrating between the limits, we find as the measure of the effect,

$$2\pi^2 \frac{v}{\lambda} a^2,$$

which is proportional to the square of the coefficient.